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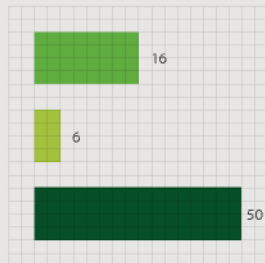
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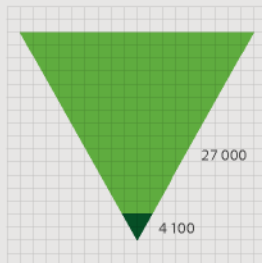
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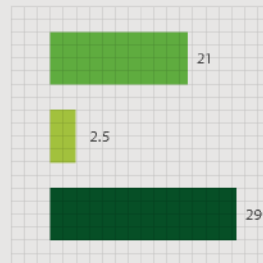
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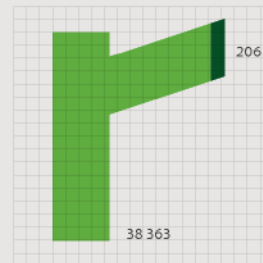
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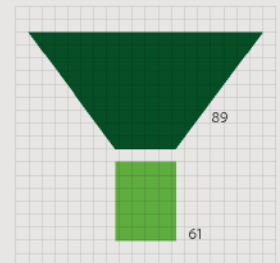
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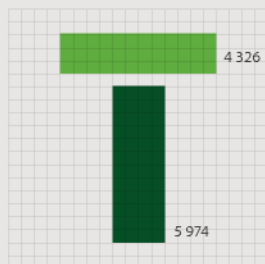
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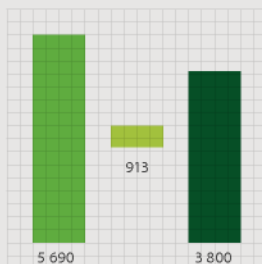
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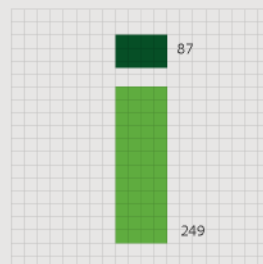
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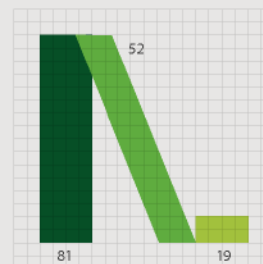
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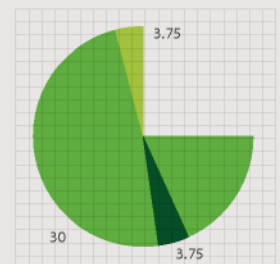
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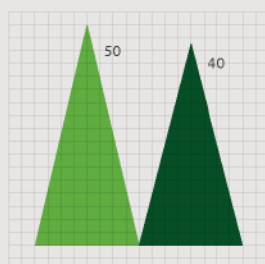
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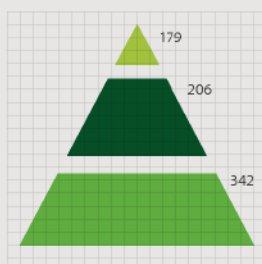
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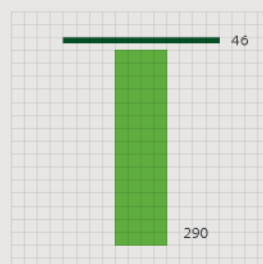
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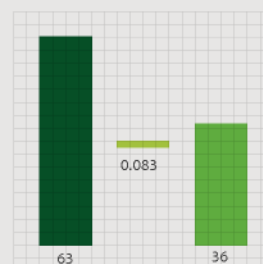
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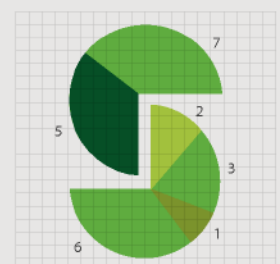
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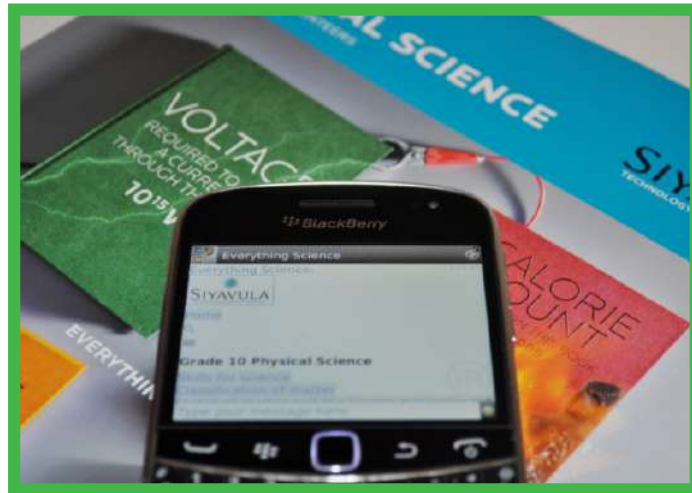
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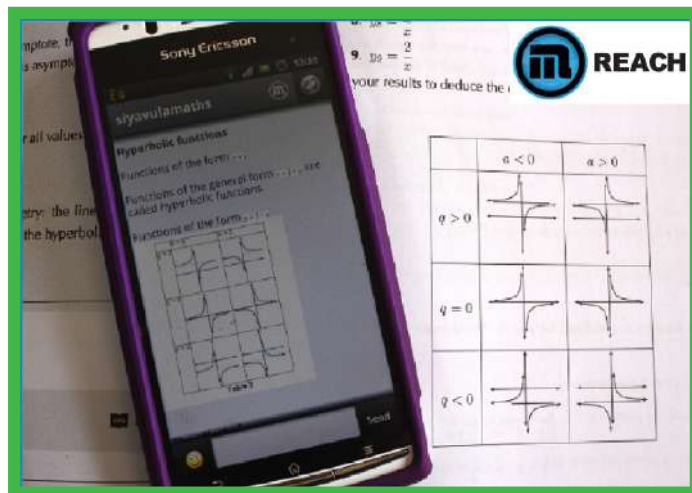
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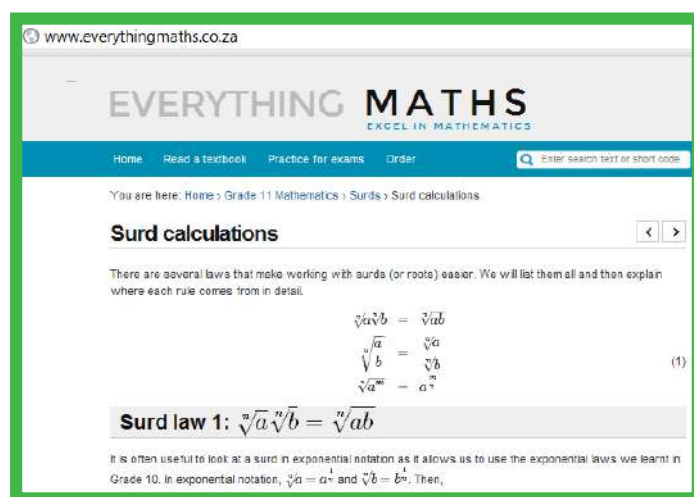
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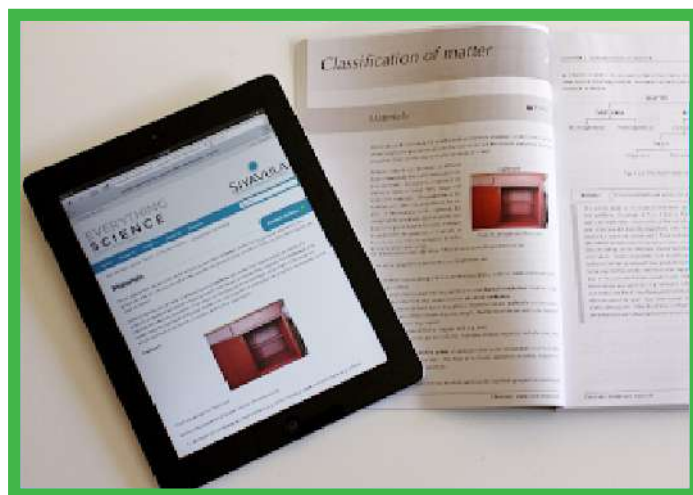
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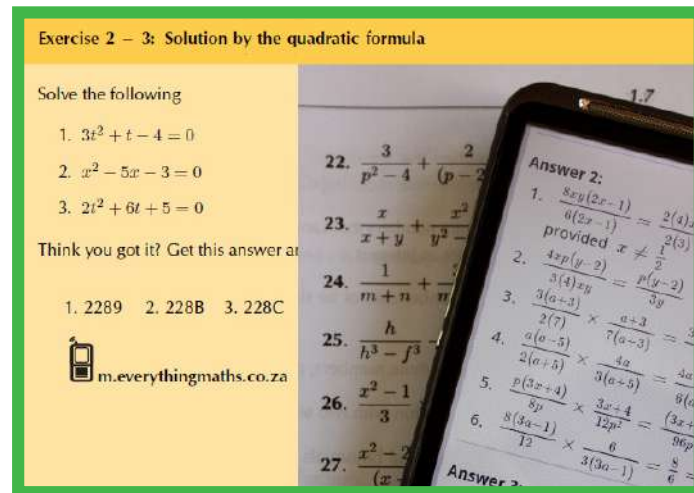
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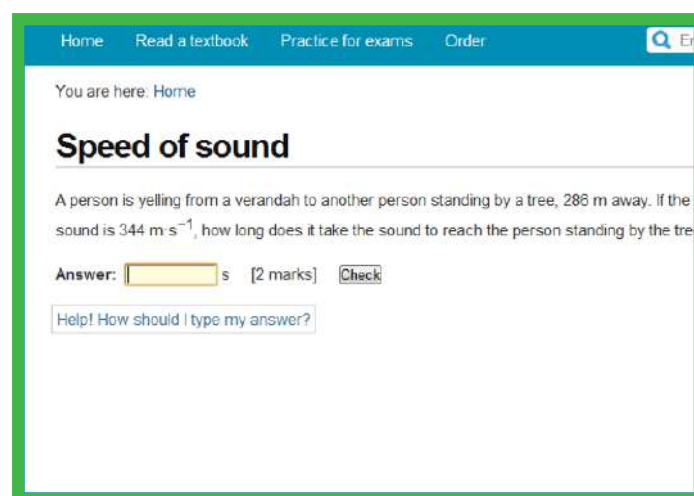
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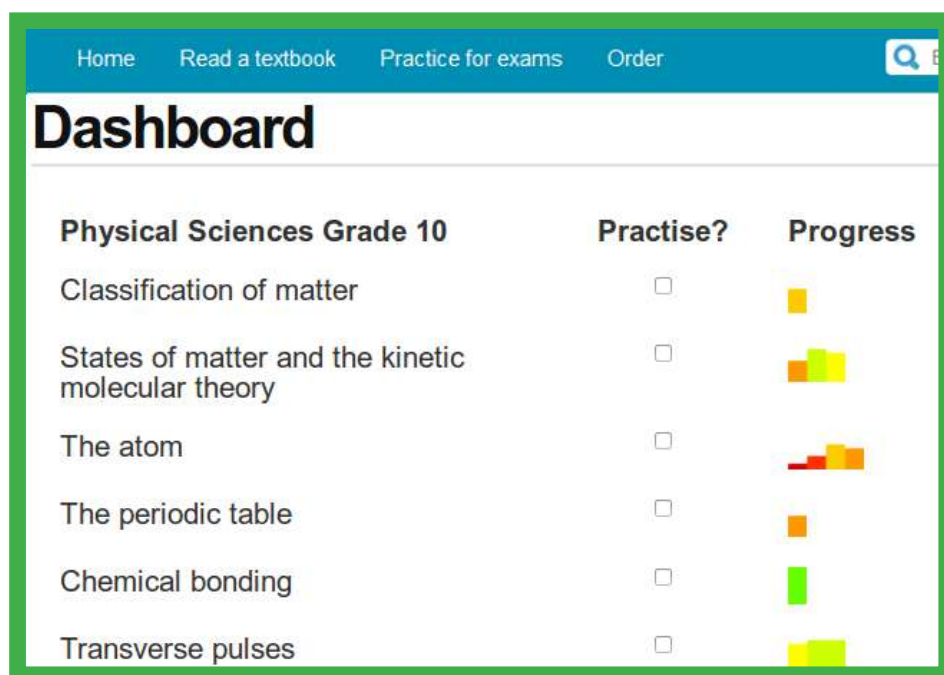

EVERYTHING MATHS

Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

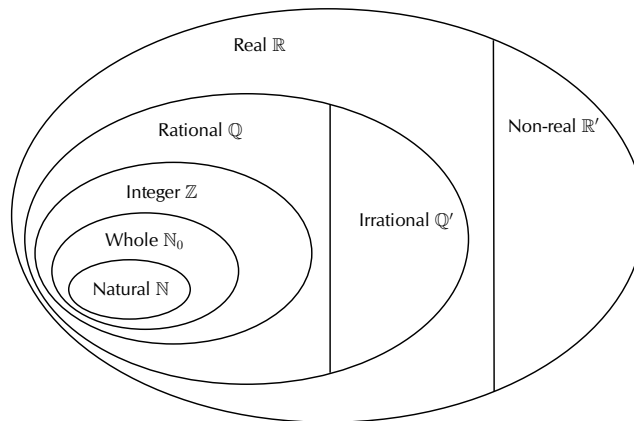
Contents

CHAPTER 1

Exponents and surds

1.1 Revision EMBF2

The number system EMBF3

The diagram below shows the structure of the number system:

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We use the following definitions:

- $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
- $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$, or as a terminating or recurring decimal number.

  Examples: $-\frac{7}{2};\ -2{,}25;\ 0;\ \sqrt{9};\ 0{,}\dot{8};\ \frac{23}{1}$
- $\mathbb{Q}'$: irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.

  Examples: $\sqrt{3};\ \sqrt[5]{2};\ \pi;\ \frac{1+\sqrt{5}}{2};\ 1{,}27548\ldots$
- $\mathbb{R}$: real numbers include all rational and irrational numbers.
- $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.

  Examples: $\sqrt{-25};\ \sqrt[4]{-1};\ -\sqrt{-\frac{1}{16}}$

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Exercise 1 – 1: The number system

Use the list of words below to describe each of the following numbers (in some cases multiple words will be applicable):

- Natural ($\mathbb{N}$)
- Whole ($\mathbb{N}_0$)
- Integer ($\mathbb{Z}$)
- Rational ($\mathbb{Q}$)
- Irrational ($\mathbb{Q}'$)
- Real ($\mathbb{R}$)
- Non-real ($\mathbb{R}'$)

1. $\sqrt{7}$
2. 0,01
3. $16\frac{2}{5}$
4. $\sqrt{6\frac{1}{4}}$
5. 0
6. 2π
7. $-5{,}3\dot{8}$
8. $\frac{1-\sqrt{2}}{2}$
9. $-\sqrt{-3}$
10. $(\pi)^2$
11. $-\frac{9}{11}$
12. $\sqrt[3]{-8}$
13. $\frac{22}{7}$
14. 2,45897...
15. $0{,}\overline{65}$
16. $\sqrt[5]{-32}$

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1. 2224 2. 2225 3. 2226 4. 2227 5. 2228 6. 2229
7. 222B 8. 222C 9. 222D 10. 222F 11. 222G 12. 222H
13. 222J 14. 222K 15. 222M 16. 222N

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Laws of exponents EMBF4

We use exponential notation to show that a number or variable is multiplied by itself a certain number of times. The exponent, also called the index or power, indicates the number of times the multiplication is repeated.

$$\text{base} \longleftarrow a^n \longrightarrow \text{exponent/index}$$

$$a^n = a \times a \times a \times \ldots \times a \quad (n \text{ times}) \qquad (a \in \mathbb{R}, n \in \mathbb{N})$$

See video: 222P at www.everythingmaths.co.za

Examples:

1. $2 \times 2 \times 2 \times 2 = 2^4$
2. $0{,}71 \times 0{,}71 \times 0{,}71 = (0{,}71)^3$
3. $(501)^2 = 501 \times 501$
4. $k^6 = k \times k \times k \times k \times k \times k$

For x^2, we say x is squared and for y^3, we say that y is cubed. In the last example we have k^6; we say that k is raised to the sixth power.

We also have the following definitions for exponents. It is important to remember that we always write the final answer with a positive exponent.

- $a^0 = 1$ ($a \neq 0$ because 0^0 is undefined)
- $a^{-n} = \frac{1}{a^n}$ ($a \neq 0$ because $\frac{1}{0}$ is undefined)

Examples:

1. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$
2. $(-36)^0 x = (1)x = x$
3. $\dfrac{7p^{-1}}{q^3 t^{-2}} = \dfrac{7t^2}{pq^3}$

We use the following laws for working with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

Worked example 1: Laws of exponents

QUESTION

Simplify the following:

1. $5(m^{2t})^p \times 2(m^{3p})^t$
2. $\dfrac{8k^3x^2}{(xk)^2}$

3. $\dfrac{2^2 \times 3 \times 7^4}{(7 \times 2)^4}$

4. $3(3^b)^a$

SOLUTION

1. $5(m^{2t})^p \times 2(m^{3p})^t = 10m^{2pt+3pt} = 10m^{5pt}$
2. $\dfrac{8k^3x^2}{(xk)^2} = \dfrac{8k^3x^2}{x^2k^2} = 8k^{(3-2)}x^{(2-2)} = 8k^1x^0 = 8k$
3. $\dfrac{2^2 \times 3 \times 7^4}{(7 \times 2)^4} = \dfrac{2^2 \times 3 \times 7^4}{7^4 \times 2^4} = 2^{(2-4)} \times 3 \times 7^{(4-4)} = 2^{-2} \times 3 = \frac{3}{4}$
4. $3(3^b)^a = 3 \times 3^{ab} = 3^{ab+1}$

Worked example 2: Laws of exponents

QUESTION

Simplify: $\dfrac{3^m - 3^{m+1}}{4 \times 3^m - 3^m}$

SOLUTION

Step 1: Simplify to a form that can be factorised

$$\frac{3^m - 3^{m+1}}{4 \times 3^m - 3^m} = \frac{3^m - (3^m \times 3)}{4 \times 3^m - 3^m}$$

Step 2: Take out a common factor

$$= \frac{3^m(1-3)}{3^m(4-1)}$$

Step 3: Cancel the common factor and simplify

$$= \frac{1-3}{4-1}$$
$$= -\frac{2}{3}$$

Exercise 1 – 2: Laws of exponents

Simplify the following:

1. $4 \times 4^{2a} \times 4^2 \times 4^a$
2. $\frac{3^2}{2^{-3}}$
3. $(3p^5)^2$
4. $\frac{k^2 k^{3x-4}}{k^x}$
5. $(5^{z-1})^2 + 5^z$
6. $(\frac{1}{4})^0$
7. $(x^2)^5$
8. $(\frac{a}{b})^{-2}$
9. $(m+n)^{-1}$
10. $2(p^t)^s$
11. $\frac{1}{\left(\frac{1}{a}\right)^{-1}}$
12. $\frac{k^0}{k^{-1}}$
13. $\frac{-2}{-2^{-a}}$
14. $\frac{-h}{(-h)^{-3}}$
15. $\left(\frac{a^2b^3}{c^3d}\right)^2$
16. $10^7(7^0) \times 10^{-6}(-6)^0 - 6$
17. $m^3n^2 \div nm^2 \times \frac{mn}{2}$
18. $(2^{-2} - 5^{-1})^{-2}$
19. $(y^2)^{-3} \div \left(\frac{x^2}{y^3}\right)^{-1}$
20. $\frac{2^{c-5}}{2^{c-8}}$
21. $\frac{2^{9a} \times 4^{6a} \times 2^2}{8^{5a}}$
22. $\frac{20t^5p^{10}}{10t^4p^9}$
23. $\left(\frac{9q^{-2s}}{q^{-3s}y^{-4a-1}}\right)^2$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 222R	2. 222S	3. 222T	4. 222V	5. 222W	6. 222X
7. 222Y	8. 222Z	9. 2232	10. 2233	11. 2234	12. 2235
13. 2236	14. 2237	15. 2238	16. 2239	17. 223B	18. 223C
19. 223D	20. 223F	21. 223G	22. 223H	23. 223J	

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See video: 222Q at www.everythingmaths.co.za

1.2 Rational exponents and surds EMBF5

The laws of exponents can also be extended to include the rational numbers. A rational number is any number that can be written as a fraction with an integer in the numerator and in the denominator. We also have the following definitions for working with rational exponents.

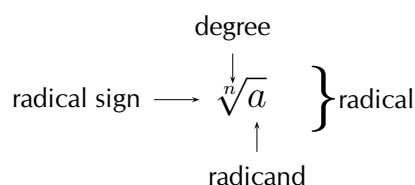
- If $r^n = a$, then $r = \sqrt[n]{a} \quad (n \geq 2)$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\frac{1}{a}}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$.

For $\sqrt{25} = 5$, we say that 5 is the square root of 25 and for $\sqrt[3]{8} = 2$, we say that 2 is the cube root of 8. For $\sqrt[5]{32} = 2$, we say that 2 is the fifth root of 32.

When dealing with exponents, a root refers to a number that is repeatedly multiplied by itself a certain number of times to get another number. A radical refers to a number written as shown below.

$$\text{radical sign} \longrightarrow \sqrt[n]{a} \quad \Big\}\text{radical}$$

degree (the n); radicand (the a)

See video: 223K at www.everythingmaths.co.za

The radical symbol and degree show which root is being determined. The radicand is the number under the radical symbol.

- If n is an even natural number, then the radicand must be positive, otherwise the roots are not real. For example, $\sqrt[4]{16} = 2$ since $2 \times 2 \times 2 \times 2 = 16$, but the roots of $\sqrt[4]{-16}$ are not real since $(-2) \times (-2) \times (-2) \times (-2) \neq -16$.
- If n is an odd natural number, then the radicand can be positive or negative. For example, $\sqrt[3]{27} = 3$ since $3 \times 3 \times 3 = 27$ and we can also determine $\sqrt[3]{-27} = -3$ since $(-3) \times (-3) \times (-3) = -27$.

It is also possible for there to be more than one n^{th} root of a number. For example, $(-2)^2 = 4$ and $2^2 = 4$, so both -2 and 2 are square roots of 4.

A surd is a radical which results in an irrational number. Irrational numbers are numbers that cannot be written as a fraction with the numerator and the denominator as integers. For example, $\sqrt{12}$, $\sqrt[3]{100}$, $\sqrt[5]{25}$ are surds.

Worked example 3: Rational exponents

QUESTION

Write each of the following as a radical and simplify where possible:

1. $18^{\frac{1}{2}}$
2. $(-125)^{-\frac{1}{3}}$

3. $4^{\frac{3}{2}}$
4. $(-81)^{\frac{1}{2}}$
5. $(0{,}008)^{\frac{1}{3}}$

SOLUTION

1. $18^{\frac{1}{2}} = \sqrt{18}$
2. $(-125)^{-\frac{1}{3}} = \sqrt[3]{(-125)^{-1}} = \sqrt[3]{\frac{1}{-125}} = \sqrt[3]{\frac{1}{(-5)^3}} = -\frac{1}{5}$
3. $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt{4^3} = \sqrt{64} = 8$
4. $(-81)^{\frac{1}{2}} = \sqrt{-81} = \text{not real}$
5. $(0{,}008)^{\frac{1}{3}} = \sqrt[3]{\frac{8}{1000}} = \sqrt[3]{\frac{2^3}{10^3}} = \frac{2}{10} = \frac{1}{5}$

See video: 223M at www.everythingmaths.co.za

Worked example 4: Rational exponents

QUESTION

Simplify without using a calculator:

$$\left(\frac{5}{4^{-1} - 9^{-1}}\right)^{\frac{1}{2}}$$

SOLUTION

Step 1: Write the fraction with positive exponents in the denominator

$$\left(\frac{5}{\frac{1}{4} - \frac{1}{9}}\right)^{\frac{1}{2}}$$

Step 2: Simplify the denominator

$$= \left(\frac{5}{\frac{9-4}{36}}\right)^{\frac{1}{2}}$$
$$= \left(\frac{5}{\frac{5}{36}}\right)^{\frac{1}{2}}$$
$$= \left(5 \div \frac{5}{36}\right)^{\frac{1}{2}}$$
$$= \left(5 \times \frac{36}{5}\right)^{\frac{1}{2}}$$
$$= (36)^{\frac{1}{2}}$$

Step 3: Take the square root

$$= \sqrt{36}$$
$$= 6$$

Exercise 1 – 3: Rational exponents and surds

1. Simplify the following and write answers with positive exponents:

   a) $\sqrt{49}$
   b) $\sqrt{36^{-1}}$
   c) $\sqrt[3]{6^{-2}}$
   d) $\sqrt[3]{-\frac{64}{27}}$
   e) $\sqrt[4]{(16x^4)^3}$

2. Simplify:

   a) $s^{\frac{1}{2}} \div s^{\frac{1}{3}}$
   b) $\left(64m^6\right)^{\frac{2}{3}}$
   c) $\dfrac{12m^{\frac{7}{9}}}{8m^{-\frac{11}{9}}}$
   d) $(5x)^0 + 5x^0 - (0{,}25)^{-0{,}5} + 8^{\frac{2}{3}}$

3. Use the laws to re-write the following expression as a power of x:
   $x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 223P 1b. 223Q 1c. 223R 1d. 223S 1e. 223T 2a. 223V
2b. 223W 2c. 223X 2d. 223Y 3. 223Z

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Simplification of surds EMBF6

We have seen in previous examples and exercises that rational exponents are closely related to surds. It is often useful to write a surd in exponential notation as it allows us to use the exponential laws.

The additional laws listed below make simplifying surds easier:

- $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
- $(\sqrt[n]{a})^m = a^{\frac{m}{n}}$

See video: 223N at www.everythingmaths.co.za

Worked example 5: Simplifying surds

QUESTION

Show that:

1. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

SOLUTION

1.

$$\begin{aligned}\sqrt[n]{a} \times \sqrt[n]{b} &= a^{\frac{1}{n}} \times b^{\frac{1}{n}} \\ &= (ab)^{\frac{1}{n}} \\ &= \sqrt[n]{ab}\end{aligned}$$

2.

$$\begin{aligned}\sqrt[n]{\frac{a}{b}} &= \left(\frac{a}{b}\right)^{\frac{1}{n}} \\ &= \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \\ &= \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\end{aligned}$$

Examples:

1. $\sqrt{2} \times \sqrt{32} = \sqrt{2 \times 32} = \sqrt{64} = 8$
2. $\dfrac{\sqrt[3]{24}}{\sqrt[3]{3}} = \sqrt[3]{\dfrac{24}{3}} = \sqrt[3]{8} = 2$
3. $\sqrt{\sqrt{81}} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

Like and unlike surds EMBF7

Two surds $\sqrt[m]{a}$ and $\sqrt[n]{b}$ are like surds if $m = n$, otherwise they are called unlike surds. For example, $\sqrt{\frac{1}{3}}$ and $-\sqrt{61}$ are like surds because $m = n = 2$. Examples of unlike surds are $\sqrt[3]{5}$ and $\sqrt[5]{7y^3}$ since $m \neq n$.

Simplest surd form EMBF8

We can sometimes simplify surds by writing the radicand as a product of factors that can be further simplified using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

See video: 2242 at www.everythingmaths.co.za

Worked example 6: Simplest surd form

QUESTION

Write the following in simplest surd form: $\sqrt{50}$

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned} \sqrt{50} &= \sqrt{5 \times 5 \times 2} \\ &= \sqrt{5^2 \times 2} \end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned} &= \sqrt{5^2} \times \sqrt{2} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2} \end{aligned}$$

Sometimes a surd cannot be simplified. For example, $\sqrt{6}$, $\sqrt[3]{30}$ and $\sqrt[4]{42}$ are already in their simplest form.

Worked example 7: Simplest surd form

QUESTION

Write the following in simplest surd form: $\sqrt[3]{54}$

SOLUTION

Step 1: Write the radicand as a product of prime factors

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{3 \times 3 \times 3 \times 2} \\ &= \sqrt[3]{3^3 \times 2}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \sqrt[3]{3^3} \times \sqrt[3]{2} \\ &= 3 \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2}\end{aligned}$$

Worked example 8: Simplest surd form

QUESTION

Simplify: $\sqrt{147} + \sqrt{108}$

SOLUTION

Step 1: Write the radicands as a product of prime factors

$$\begin{aligned}\sqrt{147} + \sqrt{108} &= \sqrt{49 \times 3} + \sqrt{36 \times 3} \\ &= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$
\begin{aligned}
&= \left(\sqrt{7^2} \times \sqrt{3}\right) + \left(\sqrt{6^2} \times \sqrt{3}\right) \\
&= \left(7 \times \sqrt{3}\right) + \left(6 \times \sqrt{3}\right) \\
&= 7\sqrt{3} + 6\sqrt{3}
\end{aligned}
$$

Step 3: Simplify and write the final answer

$$13\sqrt{3}$$

Worked example 9: Simplest surd form

QUESTION

Simplify: $\left(\sqrt{20} - \sqrt{5}\right)^2$

SOLUTION

Step 1: Factorise the radicands were possible

$$\left(\sqrt{20} - \sqrt{5}\right)^2 = \left(\sqrt{4 \times 5} - \sqrt{5}\right)^2$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$
\begin{aligned}
&= \left(\sqrt{4} \times \sqrt{5} - \sqrt{5}\right)^2 \\
&= \left(2 \times \sqrt{5} - \sqrt{5}\right)^2 \\
&= \left(2\sqrt{5} - \sqrt{5}\right)^2
\end{aligned}
$$

Step 3: Simplify and write the final answer

$$
\begin{aligned}
&= \left(\sqrt{5}\right)^2 \\
&= 5
\end{aligned}
$$

Worked example 10: Simplest surd form with fractions

QUESTION

Write in simplest surd form: $\sqrt{75} \times \sqrt[3]{(48)^{-1}}$

SOLUTION

Step 1: Factorise the radicands were possible

$$\begin{aligned}\sqrt{75} \times \sqrt[3]{(48)^{-1}} &= \sqrt{25 \times 3} \times \sqrt[3]{\frac{1}{48}} \\ &= \sqrt{25 \times 3} \times \frac{1}{\sqrt[3]{8 \times 6}}\end{aligned}$$

Step 2: Simplify using $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$\begin{aligned}&= \sqrt{25} \times \sqrt{3} \times \frac{1}{\sqrt[3]{8} \times \sqrt[3]{6}} \\ &= 5 \times \sqrt{3} \times \frac{1}{2 \times \sqrt[3]{6}}\end{aligned}$$

Step 3: Simplify and write the final answer

$$\begin{aligned}&= 5\sqrt{3} \times \frac{1}{2\sqrt[3]{6}} \\ &= \frac{5\sqrt{3}}{2\sqrt[3]{6}}\end{aligned}$$

Exercise 1 – 4: Simplification of surds

1. Simplify the following and write answers with positive exponents:

   a) $\sqrt[3]{16} \times \sqrt[3]{4}$

   b) $\sqrt{a^2b^3} \times \sqrt{b^5c^4}$

   c) $\frac{\sqrt{12}}{\sqrt{3}}$

   d) $\sqrt{x^2y^{13}} \div \sqrt{y^5}$

2. Simplify the following:

a) $\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$

b) $\frac{b-a}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$

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1a. 2243 1b. 2244 1c. 2245 1d. 2246 2a. 2247 2b. 2248

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Rationalising denominators EMBF9

It is often easier to work with fractions that have rational denominators instead of surd denominators. By rationalising the denominator, we convert a fraction with a surd in the denominator to a fraction that has a rational denominator.

Worked example 11: Rationalising the denominator

QUESTION

Rationalise the denominator:

$$\frac{5x-16}{\sqrt{x}}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{x}}{\sqrt{x}}$

Notice that $\frac{\sqrt{x}}{\sqrt{x}} = 1$, so the value of the fraction has not been changed.

$$\frac{5x-16}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}(5x-16)}{\sqrt{x} \times \sqrt{x}}$$

Step 2: Simplify the denominator

$$= \frac{\sqrt{x}(5x-16)}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x}(5x-16)}{x}$$

The term in the denominator has changed from a surd to a rational number. Expressing the surd in the numerator is the preferred way of writing expressions.

Worked example 12: Rationalising the denominator

QUESTION

Write the following with a rational denominator:

$$\frac{y - 25}{\sqrt{y} + 5}$$

SOLUTION

Step 1: Multiply the fraction by $\frac{\sqrt{y}-5}{\sqrt{y}-5}$

To eliminate the surd from the denominator, we must multiply the fraction by an expression that will result in a difference of two squares in the denominator.

$$\frac{y - 25}{\sqrt{y} + 5} \times \frac{\sqrt{y} - 5}{\sqrt{y} - 5}$$

Step 2: Simplify the denominator

$$\begin{aligned}
&= \frac{(y - 25)(\sqrt{y} - 5)}{(\sqrt{y} + 5)(\sqrt{y} - 5)} \\
&= \frac{(y - 25)(\sqrt{y} - 5)}{(\sqrt{y})^2 - 25} \\
&= \frac{(y - 25)(\sqrt{y} - 5)}{y - 25} \\
&= \sqrt{y} - 5
\end{aligned}$$

See video: 2249 at www.everythingmaths.co.za

Exercise 1 – 5: Rationalising the denominator

Rationalise the denominator in each of the following:

1. $\frac{10}{\sqrt{5}}$
2. $\frac{3}{\sqrt{6}}$
3. $\frac{2}{\sqrt{3}} \div \frac{\sqrt{2}}{3}$
4. $\frac{3}{\sqrt{5} - 1}$
5. $\frac{x}{\sqrt{y}}$
6. $\frac{\sqrt{3} + \sqrt{7}}{\sqrt{2}}$
7. $\frac{3\sqrt{p} - 4}{\sqrt{p}}$
8. $\frac{t - 4}{\sqrt{t} + 2}$
9. $(1 + \sqrt{m})^{-1}$
10. $a\left(\sqrt{a} \div \sqrt{b}\right)^{-1}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 224B 2. 224C 3. 224D 4. 224F 5. 224G 6. 224H
7. 224J 8. 224K 9. 224M 10. 224N

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1.3 Solving surd equations EMBFB

We also need to be able to solve equations that involve surds.

See video: 224P at www.everythingmaths.co.za

Worked example 13: Surd equations

QUESTION

Solve for x: $5\sqrt[3]{x^4} = 405$

SOLUTION

Step 1: Write in exponential notation

$$\begin{aligned} 5\left(x^4\right)^{\frac{1}{3}} &= 405 \\ 5x^{\frac{4}{3}} &= 405 \end{aligned}$$

Step 2: Divide both sides of the equation by 5 and simplify

$$\begin{aligned} \frac{5x^{\frac{4}{3}}}{5} &= \frac{405}{5} \\ x^{\frac{4}{3}} &= 81 \\ x^{\frac{4}{3}} &= 3^4 \end{aligned}$$

Step 3: Simplify the exponents

$$\begin{aligned} \left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} &= \left(3^4\right)^{\frac{3}{4}} \\ x &= 3^3 \\ x &= 27 \end{aligned}$$

Step 4: Check the solution by substituting the answer back into the original equation

$$\begin{aligned}
\text{LHS} &= 5\sqrt[3]{x^4} \\
&= 5(27)^{\frac{4}{3}} \\
&= 5(3^3)^{\frac{4}{3}} \\
&= 5(3^4) \\
&= 405 \\
&= \text{RHS}
\end{aligned}$$

Worked example 14: Surd equations

QUESTION

Solve for z: $z - 4\sqrt{z} + 3 = 0$

SOLUTION

Step 1: Factorise

$$z - 4\sqrt{z} + 3 = 0$$

$$z - 4z^{\frac{1}{2}} + 3 = 0$$

$$(z^{\frac{1}{2}} - 3)(z^{\frac{1}{2}} - 1) = 0$$

Step 2: Solve for both factors

The zero law states: if $a \times b = 0$, then $a = 0$ or $b = 0$.

$$\therefore (z^{\frac{1}{2}} - 3) = 0 \text{ or } (z^{\frac{1}{2}} - 1) = 0$$

Therefore

$$\begin{aligned}
z^{\frac{1}{2}} - 3 &= 0 \\
z^{\frac{1}{2}} &= 3 \\
\left(z^{\frac{1}{2}}\right)^2 &= 3^2 \\
z &= 9
\end{aligned}$$

or

$$
\begin{aligned}
z^{\frac{1}{2}} - 1 &= 0 \\
z^{\frac{1}{2}} &= 1 \\
\left(z^{\frac{1}{2}}\right)^2 &= 1^2 \\
z &= 1
\end{aligned}
$$

Step 3: Check the solution by substituting both answers back into the original equation

If $z = 9$:

$$
\begin{aligned}
\text{LHS} &= z - 4\sqrt{z} + 3 \\
&= 9 - 4\sqrt{9} + 3 \\
&= 12 - 12 \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

If $z = 1$:

$$
\begin{aligned}
\text{LHS} &= z - 4\sqrt{z} + 3 \\
&= 1 - 4\sqrt{1} + 3 \\
&= 4 - 4 \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

Step 4: Write the final answer

The solution to $z - 4\sqrt{z} + 3 = 0$ is $z = 9$ or $z = 1$.

Worked example 15: Surd equations

QUESTION

Solve for p: $\sqrt{p-2} - 3 = 0$

SOLUTION

Step 1: Write the equation with only the square root on the left hand side

Use the additive inverse to get all other terms on the right hand side and only the

square root on the left hand side.

$$\sqrt{p-2} = 3$$

Step 2: Square both sides of the equation

$$\left(\sqrt{p-2}\right)^2 = 3^2$$
$$p - 2 = 9$$
$$p = 11$$

Step 3: Check the solution by substituting the answer back into the original equation

If $p = 11$:

$$\begin{aligned} \text{LHS} &= \sqrt{p-2} - 3 \\ &= \sqrt{11-2} - 3 \\ &= \sqrt{9} - 3 \\ &= 3 - 3 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Step 4: Write the final answer

The solution to $\sqrt{p-2} - 3 = 0$ is $p = 11$.

Exercise 1 – 6: Solving surd equations

Solve for the unknown variable (remember to check that the solution is valid):

1. $2^{x+1} - 32 = 0$
2. $125\left(3^p\right) = 27\left(5^p\right)$
3. $2y^{\frac{1}{2}} - 3y^{\frac{1}{4}} + 1 = 0$
4. $t - 1 = \sqrt{7-t}$
5. $2z - 7\sqrt{z} + 3 = 0$
6. $x^{\frac{1}{3}}(x^{\frac{1}{3}} + 1) = 6$
7. $2^{4n} - \dfrac{1}{\sqrt[4]{16}} = 0$
8. $\sqrt{31 - 10d} = 4 - d$
9. $y - 10\sqrt{y} + 9 = 0$
10. $f = 2 + \sqrt{19 - 2f}$

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1. 224Q 2. 224R 3. 224S 4. 224T 5. 224V 6. 224W
7. 224X 8. 224Y 9. 224Z 10. 2252

1.4 Applications of exponentials EMBFC

There are many real world applications that require exponents. For example, exponentials are used to determine population growth and they are also used in finance to calculate different types of interest.

Worked example 16: Applications of exponentials

QUESTION

A type of bacteria has a very high exponential growth rate at 80% every hour. If there are 10 bacteria, determine how many there will be in five hours, in one day and in one week?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in hours}}$$

Therefore, in this case:

$$\text{final population} = 10\,(1{,}8)^{n}$$

where n = number of hours.

Step 2: In 5 hours

$\text{final population} = 10\,(1{,}8)^{5} \approx 189$

Step 3: In 1 day = 24 hours

$\text{final population} = 10\,(1{,}8)^{24} \approx 13\ 382\ 588$

Step 4: In 1 week = 168 hours

$\text{final population} = 10\,(1{,}8)^{168} \approx 7{,}687 \times 10^{43}$

Note this answer is given in scientific notation as it is a very big number.

Worked example 17: Applications of exponentials

QUESTION

A species of extremely rare deep water fish has a very long lifespan and rarely has offspring. If there are a total of 821 of this type of fish and their growth rate is 2% each month, how many will there be in half of a year? What will the population be in ten years and in one hundred years?

SOLUTION

Step 1: Exponential formula

$$\text{final population} = \text{initial population} \times (1 + \text{growth percentage})^{\text{time period in months}}$$

Therefore, in this case:

$$\text{final population} = 821(1{,}02)^n$$

where n = number of months.

Step 2: In half a year = 6 months

final population $= 821(1{,}02)^6 \approx 925$

Step 3: In 10 years = 120 months

final population $= 821(1{,}02)^{120} \approx 8838$

Step 4: In 100 years = 1200 months

final population $= 821(1{,}02)^{1200} \approx 1{,}716 \times 10^{13}$

Note this answer is also given in scientific notation as it is a very big number.

Exercise 1 – 7: Applications of exponentials

1. Nqobani invests R 5530 into an account which pays out a lump sum at the end of 6 years. If he gets R 9622,20 at the end of the period, what compound interest rate did the bank offer him? Give answer correct to one decimal place.
2. The current population of Johannesburg is 3 885 840 and the average rate of population growth in South Africa is 0,7% p.a. What can city planners expect the population of Johannesburg to be in 13 years time?
3. Abiona places 3 books in a stack on her desk. The next day she counts the books in the stack and then adds the same number of books to the top of the stack. After how many days will she have a stack of 192 books?

4. A type of mould has a very high exponential growth rate of 40% every hour. If there are initially 45 individual mould cells in the population, determine how many there will be in 19 hours.

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1. 2253 2. 2254 3. 2255 4. 2256

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1.5 Summary EMBFD

See presentation: 2257 at www.everythingmaths.co.za

1. The number system:
   - $\mathbb{N}$: natural numbers are $\{1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{N}_0$: whole numbers are $\{0;\ 1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{Z}$: integers are $\{\ldots;\ -3;\ -2;\ -1;\ 0;\ 1;\ 2;\ 3;\ \ldots\}$
   - $\mathbb{Q}$: rational numbers are numbers which can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$, or as a terminating or recurring decimal number.
   - $\mathbb{Q}'$: irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers. Irrational numbers also include decimal numbers that neither terminate nor recur.
   - $\mathbb{R}$: real numbers include all rational and irrational numbers.
   - $\mathbb{R}'$: non-real numbers or imaginary numbers are numbers that are not real.
2. Definitions:
   - $a^n = a \times a \times a \times \cdots \times a \quad (n \text{ times}) \qquad (a \in \mathbb{R}, n \in \mathbb{N})$
   - $a^0 = 1 \qquad (a \neq 0$ because 0^0 is undefined)
   - $a^{-n} = \frac{1}{a^n} \qquad (a \neq 0$ because $\dfrac{1}{0}$ is undefined)
3. Laws of exponents:
   - $a^m \times a^n = a^{m+n}$
   - $\dfrac{a^m}{a^n} = a^{m-n}$
   - $(ab)^n = a^n b^n$
   - $\left(\frac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
   - $(a^m)^n = a^{mn}$

   where $a > 0$, $b > 0$ and $m, n \in \mathbb{Z}$.

4. Rational exponents and surds:

   - If $r^n = a$, then $r = \sqrt[n]{a} \quad (n \geq 2)$
   - $a^{\frac{1}{n}} = \sqrt[n]{a}$
   - $a^{-\frac{1}{n}} = (a^{-1})^{\frac{1}{n}} = \sqrt[n]{\frac{1}{a}}$
   - $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

   where $a > 0$, $r > 0$ and $m, n \in \mathbb{Z}$, $n \neq 0$.

5. Simplification of surds:

   - $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$
   - $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
   - $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Exercise 1 – 8: End of chapter exercises

1. Simplify as far as possible:

   a) $8^{-\frac{2}{3}}$

   b) $\sqrt{16} + 8^{-\frac{2}{3}}$

2. Simplify:

   a) $\left(x^3\right)^{\frac{4}{3}}$

   b) $\left(s^2\right)^{\frac{1}{2}}$

   c) $\left(m^5\right)^{\frac{5}{3}}$

   d) $\left(-m^2\right)^{\frac{4}{3}}$

   e) $-\left(m^2\right)^{\frac{4}{3}}$

   f) $\left(3y^{\frac{4}{3}}\right)^4$

3. Simplify the following:

   a) $\frac{3a^{-2}b^{15}c^{-5}}{\left(a^{-4}b^3c\right)^{\frac{-5}{2}}}$

   b) $\left(9a^6b^4\right)^{\frac{1}{2}}$

   c) $\left(a^{\frac{3}{2}}b^{\frac{3}{4}}\right)^{16}$

   d) $x^3\sqrt{x}$

   e) $\sqrt[3]{x^4b^5}$

4. Re-write the following expression as a power of x:

   $\frac{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{x^2}$

5. Expand:

   $\left(\sqrt{x} - \sqrt{2}\right)\left(\sqrt{x} + \sqrt{2}\right)$

6. Rationalise the denominator:

   $\frac{10}{\sqrt{x} - \frac{1}{x}}$

7. Write as a single term with a rational denominator:
$\frac{3}{2\sqrt{x}} + \sqrt{x}$

8. Write in simplest surd form:

   a) $\sqrt{72}$
   b) $\sqrt{45} + \sqrt{80}$
   c) $\frac{\sqrt{48}}{\sqrt{12}}$
   d) $\frac{\sqrt{18} \div \sqrt{72}}{\sqrt{8}}$
   e) $\frac{4}{(\sqrt{8} \div \sqrt{2})}$
   f) $\frac{16}{(\sqrt{20} \div \sqrt{12})}$

9. Expand and simplify:

   a) $\left(2 + \sqrt{2}\right)^2$
   b) $\left(2 + \sqrt{2}\right)\left(1 + \sqrt{8}\right)$
   c) $\left(1 + \sqrt{3}\right)\left(1 + \sqrt{8} + \sqrt{3}\right)$

10. Simplify, without use of a calculator:

   a) $\sqrt{5}\left(\sqrt{45} + 2\sqrt{80}\right)$
   b) $\frac{\sqrt{98} - \sqrt{8}}{\sqrt{50}}$

11. Simplify:
$\sqrt{98x^6} + \sqrt{128x^6}$

12. Rationalise the denominator:

   a) $\frac{\sqrt{5} + 2}{\sqrt{5}}$
   b) $\frac{y - 4}{\sqrt{y} - 2}$
   c) $\frac{2x - 20}{\sqrt{x} - \sqrt{10}}$

13. Evaluate without using a calculator: $\left(2 - \frac{\sqrt{7}}{2}\right)^{\frac{1}{2}} \times \left(2 + \frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$

14. Prove (without the use of a calculator):
$\sqrt{\frac{8}{3}} + 5\sqrt{\frac{5}{3}} - \sqrt{\frac{1}{6}} = \frac{10\sqrt{15} + 3\sqrt{6}}{6}$

15. Simplify completely by showing all your steps (do not use a calculator):

$$3^{-\frac{1}{2}}\left[\sqrt{12} + \sqrt[3]{\left(3\sqrt{3}\right)}\right]$$

16. Fill in the blank surd-form number on the right hand side of the equal sign which will make the following a true statement: $-3\sqrt{6} \times -2\sqrt{24} = -\sqrt{18} \times \ldots$

17. Solve for the unknown variable:

a) $3^{x-1} - 27 = 0$

b) $8^x - \frac{1}{\sqrt[3]{8}} = 0$

c) $27(4^x) = (64)3^x$

d) $\sqrt{2x-5} = 2 - x$

e) $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 2 = 0$

18. a) Show that $\sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3}$ is equal to 3

b) Hence solve $\sqrt{\frac{3^{x+1} - 3^x}{3^{x-1}} + 3} = \left(\frac{1}{3}\right)^{x-2}$

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1a. 2258	1b. 2259	2a. 225B	2b. 225C	2c. 225D	2d. 225F
2e. 225G	2f. 225H	3a. 225J	3b. 225K	3c. 225M	3d. 225N
3e. 225P	4. 225Q	5. 225R	6. 225S	7. 225T	8a. 225V
8b. 225W	8c. 225X	8d. 225Y	8e. 225Z	8f. 2262	9a. 2263
9b. 2264	9c. 2265	10a. 2266	10b. 2267	11. 2268	12a. 2269
12b. 226B	12c. 226C	13. 226D	14. 226F	15. 226G	16. 226H
17a. 226J	17b. 226K	17c. 226M	17d. 226N	17e. 226P	18. 226Q

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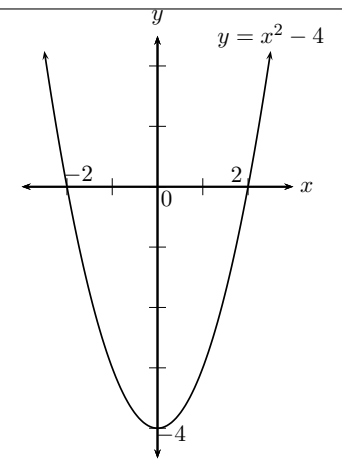
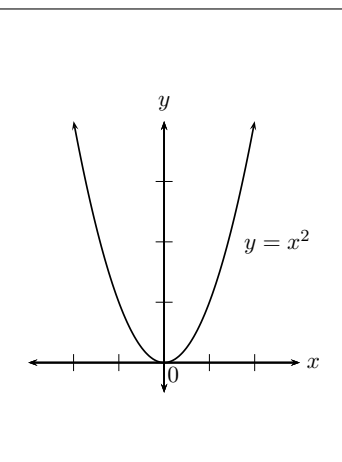
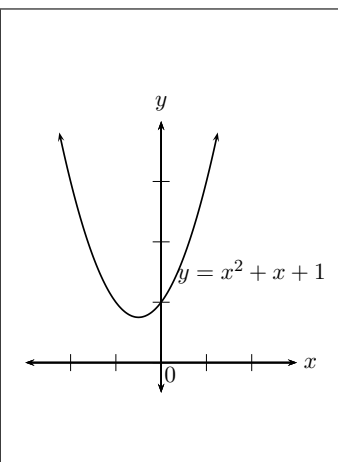
CHAPTER

Equations and inequalities

2.1 Revision EMBFF

Solving quadratic equations using factorisation EMBFG

Terminology:	
Expression	An expression is a term or group of terms consisting of numbers, variables and the basic operators $(+, -, \times, \div, x^n)$.
Equation	A mathematical statement that asserts that two expressions are equal.
Inequality	An inequality states the relation between two expressions $(>, <, \geq, \leq)$.
Solution	A value or set of values that satisfy the original problem statement.
Root	A root of an equation is the value of x such that $f(x) = 0$.

A quadratic equation is an equation of the second degree; the exponent of one variable is 2.

The following are examples of quadratic equations:

$$2x^2 - 5x = 12$$

$$a(a - 3) - 10 = 0$$

$$\frac{3b}{b+2} + 1 = \frac{4}{b+1}$$

A quadratic equation has at most two solutions, also referred to as roots. There are some situations, however, in which a quadratic equation has either one solution or no solutions.

$y = (x - 2)(x + 2)$ $= x^2 - 4$ Graph of a quadratic equation with two roots: $x = -2$ and $x = 2$.	$y = x^2$ Graph of a quadratic equation with one root: $x = 0$.	$y = x^2 + x + 1$ Graph of a quadratic equation with no real roots.

One method for solving quadratic equations is factorisation. The standard form of a quadratic equation is $ax^2 + bx + c = 0$ and it is the starting point for solving any equation by factorisation.

It is very important to note that one side of the equation must be equal to zero.

Investigation: Zero product law

Solve the following equations:

1. $6 \times 0 = ?$
2. $-25 \times 0 = ?$
3. $0 \times 0{,}69 = ?$
4. $7 \times ? = 0$

Now solve for the variable in each of the following:

1. $6 \times m = 0$
2. $32 \times x \times 2 = 0$
3. $11(z - 3) = 0$
4. $(k + 3)(k - 4) = 0$

To obtain the two roots we use the fact that if $a \times b = 0$, then $a = 0$ and/or $b = 0$. This is called the zero product law.

Method for solving quadratic equations EMBFH

1. Rewrite the equation in the standard form $ax^2 + bx + c = 0$.
2. Divide the entire equation by any common factor of the coefficients to obtain a simpler equation of the form $ax^2+bx+c = 0$, where a, b and c have no common factors.
3. Factorise $ax^2 + bx + c = 0$ to be of the form $(rx + s)(ux + v) = 0$.
4. The two solutions are

$$(rx + s) = 0 \qquad (ux + v) = 0$$
$$\text{So } x = -\frac{s}{r} \qquad \text{So } x = -\frac{v}{u}$$

5. Always check the solution by substituting the answer back into the original equation.

See video: 226R at www.everythingmaths.co.za

Worked example 1: Solving quadratic equations using factorisation

QUESTION

Solve for x: $x(x - 3) = 10$

SOLUTION

Step 1: Rewrite the equation in the form $ax^2 + bx + c = 0$

Expand the brackets and subtract 10 from both sides of the equation $x^2 - 3x - 10 = 0$

Step 2: Factorise

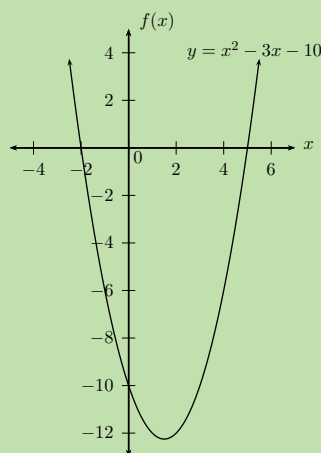
$(x + 2)(x - 5) = 0$

Step 3: Solve for both factors

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

or

$$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned}$$

The graph shows the roots of the equation $x = -2$ or $x = 5$. This graph does not form part of the answer as the question did not ask for a sketch. It is shown here for illustration purposes only.

Step 4: Check the solution by substituting both answers back into the original equation

Step 5: Write the final answer

Therefore $x = -2$ or $x = 5$.

Worked example 2: Solving quadratic equations using factorisation

QUESTION

Solve the equation: $2x^2 - 5x - 12 = 0$

SOLUTION

Step 1: There are no common factors

Step 2: The quadratic equation is already in the standard form $ax^2 + bx + c = 0$

Step 3: Factorise

We must determine the combination of factors of 2 and 12 that will give a middle term coefficient of 5.

We find that 2×1 and 3×4 give a middle term coefficient of 5 so we can factorise the equation as

$$(2x + 3)(x - 4) = 0$$

Step 4: Solve for both roots

We have

$$2x + 3 = 0$$
$$x = -\frac{3}{2}$$

or

$$x - 4 = 0$$
$$x = 4$$

Step 5: Check the solution by substituting both answers back into the original equation

Step 6: Write the final answer

Therefore, $x = -\frac{3}{2}$ or $x = 4$.

Worked example 3: Solving quadratic equations using factorisation

QUESTION

Solve for y: $y^2 - 7 = 0$

SOLUTION

Step 1: Factorise as a difference of two squares

We know that

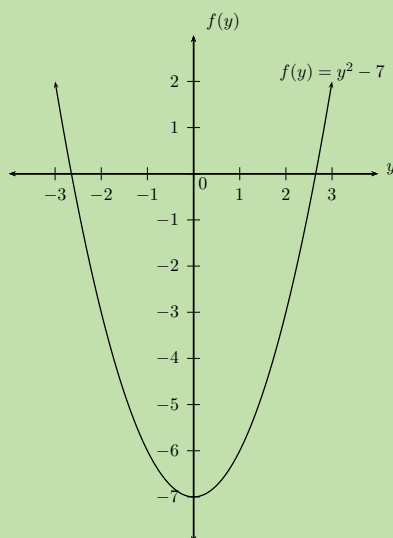
$$\left(\sqrt{7}\right)^2 = 7$$

We can write the equation as

$$y^2 - (\sqrt{7})^2 = 0$$

Step 2: Factorise

$$(y - \sqrt{7})(y + \sqrt{7}) = 0$$
$$\text{Therefore } y = \sqrt{7} \text{ or } y = -\sqrt{7}$$

Even though the question did not ask for a sketch, it is often very useful to draw the graph. We can let $f(y) = y^2 - 7$ and draw a rough sketch of the graph to see where the two roots of the equation lie.

Step 3: Check the solution by substituting both answers back into the original equation

Step 4: Write the final answer

Therefore $y = \pm\sqrt{7}$.

Worked example 4: Solving quadratic equations using factorisation

QUESTION

Solve for b:

$$\frac{3b}{b+2} + 1 = \frac{4}{b+1}$$

SOLUTION

Step 1: Determine the restrictions

The restrictions are the values for b that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $b \neq -2$ and $b \neq -1$.

Step 2: Determine the lowest common denominator

The lowest common denominator is $(b+2)(b+1)$.

Step 3: Multiply each term in the equation by the lowest common denominator and simplify

$$\frac{3b(b+2)(b+1)}{b+2} + (b+2)(b+1) = \frac{4(b+2)(b+1)}{b+1}$$

$$3b(b+1) + (b+2)(b+1) = 4(b+2)$$

$$3b^2 + 3b + b^2 + 3b + 2 = 4b + 8$$

$$4b^2 + 2b - 6 = 0$$

$$2b^2 + b - 3 = 0$$

Step 4: Factorise and solve the equation

$$(2b+3)(b-1) = 0$$

$$2b + 3 = 0 \text{ or } b - 1 = 0$$

$$b = -\frac{3}{2} \text{ or } b = 1$$

Step 5: Check the solution by substituting both answers back into the original equation

Step 6: Write the final answer

Therefore $b = -1\frac{1}{2}$ or $b = 1$.

Worked example 5: Squaring both sides of the equation

QUESTION

Solve for m: $m + 2 = \sqrt{7 + 2m}$

SOLUTION

Step 1: Square both sides of the equation

Before we square both sides of the equation, we must make sure that the radical is the only term on one side of the equation and all other terms are on the other, otherwise squaring both sides will make the equation more complicated to solve.

$$(m+2)^2 = \left(\sqrt{7+2m}\right)^2$$

Step 2: Expand the brackets and simplify

$$\begin{aligned}(m+2)^2 &= \left(\sqrt{7+2m}\right)^2\\ m^2+4m+4 &= 7+2m\\ m^2+2m-3 &= 0\end{aligned}$$

Step 3: Factorise and solve for m

$$(m-1)(m+3)=0$$
$$\text{Therefore } m=1 \text{ or } m=-3$$

Step 4: Check the solution by substituting both answers back into the original equation

To find the solution we squared both sides of the equation. Squaring an expression changes negative values to positives and can therefore introduce invalid answers into the solution. Therefore it is very important to check that the answers obtained are valid. To test the answers, always substitute back into the original equation.

If $m=1$:

$$\begin{aligned}\text{RHS} &= \sqrt{7+2(1)}\\ &= \sqrt{9}\\ &= 3\\ \text{LHS} &= 1+2\\ &= 3\\ \text{LHS} &= \text{RHS}\end{aligned}$$

Therefore $m=1$ is valid.

If $m=-3$:

$$\begin{aligned}\text{RHS} &= \sqrt{7+2(-3)}\\ &= \sqrt{1}\\ &= 1\\ \text{LHS} &= -3+2\\ &= -1\\ \text{LHS} &\neq \text{RHS}\end{aligned}$$

Therefore $m=-3$ is not valid.

Step 5: Write the final answer

Therefore $m=1$.

See video: 226S at www.everythingmaths.co.za

Exercise 2 – 1: Solution by factorisation

Solve the following quadratic equations by factorisation. Answers may be left in surd form, where applicable.

1. $7t^2 + 14t = 0$
2. $12y^2 + 24y + 12 = 0$
3. $16s^2 = 400$
4. $y^2 - 5y + 6 = 0$
5. $y^2 + 5y - 36 = 0$
6. $4 + p = \sqrt{p + 6}$
7. $-y^2 - 11y - 24 = 0$
8. $13y - 42 = y^2$
9. $(x - 1)(x + 10) = -24$
10. $y^2 - 5ky + 4k^2 = 0$
11. $2y^2 - 61 = 101$
12. $2y^2 - 10 = 0$
13. $-8 + h^2 = 28$
14. $y^2 - 4 = 10$
15. $\sqrt{5 - 2p} - 4 = \frac{1}{2}p$
16. $y^2 + 28 = 100$
17. $f(2f + 1) = 15$
18. $2x = \sqrt{21x - 5}$
19. $\frac{5y}{y - 2} + \frac{3}{y} + 2 = \frac{-6}{y^2 - 2y}$
20. $\frac{x + 9}{x^2 - 9} + \frac{1}{x + 3} = \frac{2}{x - 3}$
21. $\frac{y - 2}{y + 1} = \frac{2y + 1}{y - 7}$
22. $1 + \frac{t - 2}{t - 1} = \frac{5}{t^2 - 4t + 3} + \frac{10}{3 - t}$
23. $\frac{4}{m + 3} + \frac{4}{4 - m^2} = \frac{5m - 5}{m^2 + m - 6}$
24. $5\sqrt{5t + 1} - 4 = 5t + 1$

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1. 226T	2. 226V	3. 226W	4. 226X	5. 226Y	6. 226Z
7. 2272	8. 2273	9. 2274	10. 2275	11. 2276	12. 2277
13. 2278	14. 2279	15. 227B	16. 227C	17. 227D	18. 227F
19. 227G	20. 227H	21. 227J	22. 227K	23. 227M	24. 227N

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2.2 Completing the square EMBFJ

Investigation: Completing the square

Can you solve each equation using two different methods?

1. $x^2 - 4 = 0$
2. $x^2 - 8 = 0$
3. $x^2 - 4x + 4 = 0$

4. $x^2 - 4x - 4 = 0$

Factorising the last equation is quite difficult. Use the previous examples as a hint and try to create a difference of two squares.

See video: 227P at www.everythingmaths.co.za

We have seen that expressions of the form $x^2 - b^2$ are known as differences of squares and can be factorised as $(x - b)(x + b)$. This simple factorisation leads to another technique for solving quadratic equations known as completing the square.

Consider the equation $x^2 - 2x - 1 = 0$. We cannot easily factorise this expression. When we expand the perfect square $(x - 1)^2$ and examine the terms we see that $(x - 1)^2 = x^2 - 2x + 1$.

We compare the two equations and notice that only the constant terms are different. We can create a perfect square by adding and subtracting the same amount to the original equation.

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ (x^2 - 2x + 1) - 1 - 1 &= 0 \\ (x^2 - 2x + 1) - 2 &= 0 \\ (x - 1)^2 - 2 &= 0 \end{aligned}$$

Method 1: Take square roots on both sides of the equation to solve for x.

$$\begin{aligned} (x - 1)^2 - 2 &= 0 \\ (x - 1)^2 &= 2 \\ \sqrt{(x - 1)^2} &= \pm\sqrt{2} \\ x - 1 &= \pm\sqrt{2} \\ x &= 1 \pm \sqrt{2} \end{aligned}$$

Therefore $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$

Very important: Always remember to include both a positive and a negative answer when taking the square root, since $2^2 = 4$ and $(-2)^2 = 4$.

Method 2: Factorise the expression as a difference of two squares using $2 = \left(\sqrt{2}\right)^2$.

We can write

$$\begin{aligned} (x - 1)^2 - 2 &= 0 \\ (x - 1)^2 - \left(\sqrt{2}\right)^2 &= 0 \\ \left((x - 1) + \sqrt{2}\right)\left((x - 1) - \sqrt{2}\right) &= 0 \end{aligned}$$

The solution is then

$$\begin{aligned}(x-1)+\sqrt{2}&=0\\ x&=1-\sqrt{2}\end{aligned}$$

or

$$\begin{aligned}(x-1)-\sqrt{2}&=0\\ x&=1+\sqrt{2}\end{aligned}$$

Method for solving quadratic equations by completing the square

1. Write the equation in the standard form $ax^2+bx+c=0$.
2. Make the coefficient of the x^2 term equal to 1 by dividing the entire equation by a.
3. Take half the coefficient of the x term and square it; then add and subtract it from the equation so that the equation remains mathematically correct. In the example above, we added 1 to complete the square and then subtracted 1 so that the equation remained true.
4. Write the left hand side as a difference of two squares.
5. Factorise the equation in terms of a difference of squares and solve for x.

See video: 227Q at www.everythingmaths.co.za

Worked example 6: Solving quadratic equations by completing the square

QUESTION

Solve by completing the square: $x^2-10x-11=0$

SOLUTION

Step 1: The equation is already in the form $ax^2+bx+c=0$

Step 2: Make sure the coefficient of the x^2 term is equal to 1

$$x^2-10x-11=0$$

Step 3: Take half the coefficient of the x term and square it; then add and subtract it from the equation

The coefficient of the x term is -10. Half of the coefficient of the x term is -5 and the square of it is 25. Therefore $x^2-10x+25-25-11=0$.

Step 4: Write the trinomial as a perfect square

$$(x^2 - 10x + 25) - 25 - 11 = 0$$
$$(x - 5)^2 - 36 = 0$$

Step 5: Method 1: Take square roots on both sides of the equation

$$(x - 5)^2 - 36 = 0$$
$$(x - 5)^2 = 36$$
$$x - 5 = \pm\sqrt{36}$$

Important: When taking a square root always remember that there is a positive and negative answer, since $(6)^2 = 36$ and $(-6)^2 = 36$.

$$x - 5 = \pm 6$$

Step 6: Solve for x

$$x = -1 \text{ or } x = 11$$

Step 7: Method 2: Factorise equation as a difference of two squares

$$(x - 5)^2 - (6)^2 = 0$$
$$[(x - 5) + 6]\,[(x - 5) - 6] = 0$$

Step 8: Simplify and solve for x

$$(x + 1)(x - 11) = 0$$
$$\therefore x = -1 \text{ or } x = 11$$

Step 9: Write the final answer

$$x = -1 \text{ or } x = 11$$

Notice that both methods produce the same answer. These roots are rational because 36 is a perfect square.

Worked example 7: Solving quadratic equations by completing the square

QUESTION

Solve by completing the square: $2x^2 - 6x - 10 = 0$

SOLUTION

Step 1: The equation is already in standard form $ax^2 + bx + c = 0$

Step 2: Make sure that the coefficient of the x^2 term is equal to 1

The coefficient of the x^2 term is 2. Therefore divide the entire equation by 2:

$$x^2 - 3x - 5 = 0$$

Step 3: Take half the coefficient of the x term, square it; then add and subtract it from the equation

The coefficient of the x term is -3, so then $\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$:

$$\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} - 5 = 0$$

Step 4: Write the trinomial as a perfect square

$$\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{20}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{29}{4} = 0$$

Step 5: Method 1: Take square roots on both sides of the equation

$$\left(x - \frac{3}{2}\right)^2 - \frac{29}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}}$$

Remember: When taking a square root there is a positive and a negative answer.

Step 6: Solve for x

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

$$= \frac{3 \pm \sqrt{29}}{2}$$

Step 7: Method 2: Factorise equation as a difference of two squares

$$\left(x - \frac{3}{2}\right)^2 - \frac{29}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \left(\sqrt{\frac{29}{4}}\right)^2 = 0$$

$$\left(x - \frac{3}{2} - \sqrt{\frac{29}{4}}\right)\left(x - \frac{3}{2} + \sqrt{\frac{29}{4}}\right) = 0$$

Step 8: Solve for x

$$\left(x - \frac{3}{2} - \frac{\sqrt{29}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{29}}{2}\right) = 0$$

$$\text{Therefore } x = \frac{3}{2} + \frac{\sqrt{29}}{2} \text{ or } x = \frac{3}{2} - \frac{\sqrt{29}}{2}$$

Notice that these roots are irrational since 29 is not a perfect square.

See video: 227R at www.everythingmaths.co.za

Exercise 2 – 2: Solution by completing the square

1. Solve the following equations by completing the square:

   a) $x^2 + 10x - 2 = 0$
   b) $x^2 + 4x + 3 = 0$
   c) $p^2 - 5 = -8p$
   d) $2(6x + x^2) = -4$
   e) $x^2 + 5x + 9 = 0$
   f) $t^2 + 30 = 2(10 - 8t)$
   g) $3x^2 + 6x - 2 = 0$
   h) $z^2 + 8z - 6 = 0$
   i) $2z^2 = 11z$
   j) $5 + 4z - z^2 = 0$

2. Solve for k in terms of a: $k^2 + 6k + a = 0$
3. Solve for y in terms of p, q and r: $py^2 + qy + r = 0$

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1a. 227S	1b. 227T	1c. 227V	1d. 227W	1e. 227X	1f. 227Y
1g. 227Z	1h. 2282	1i. 2283	1j. 2284	2. 2285	3. 2286

2.3 Quadratic formula

EMBFK

It is not always possible to solve a quadratic equation by factorisation and it can take a long time to complete the square. The method of completing the square provides a way to derive a formula that can be used to solve any quadratic equation. The quadratic formula provides an easy and fast way to solve quadratic equations.

Consider the standard form of the quadratic equation $ax^2 + bx + c = 0$. Divide both sides by a $(a \neq 0)$ to get

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

Now using the method of completing the square, we must halve the coefficient of x and square it. We then add and subtract $\left(\frac{b}{2a}\right)^2$ so that the equation remains true.

$$\begin{aligned} x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\ \left(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\ \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} &= 0 \end{aligned}$$

We add the constant to both sides and take the square root of both sides of the equation, being careful to include a positive and negative answer.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Therefore, for any quadratic equation $ax^2+bx+c=0$ we can determine two roots

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

It is important to notice that the expression b^2-4ac must be greater than or equal to zero for the roots of the quadratic to be real. If the expression under the square root sign is less than zero, then the roots are non-real (imaginary).

See video: 2287 at www.everythingmaths.co.za

Worked example 8: Using the quadratic formula

QUESTION

Solve for x and leave your answer in simplest surd form: $2x^2+3x=7$

SOLUTION

Step 1: Check whether the expression can be factorised

The expression cannot be factorised, so the general quadratic formula must be used.

Step 2: Write the equation in the standard form $ax^2+bx+c=0$

$$2x^2+3x-7=0$$

Step 3: Identify the coefficients to substitute into the formula

$$a=2; \qquad b=3; \qquad c=-7$$

Step 4: Apply the quadratic formula

Always write down the formula first and then substitute the values of a, b and c.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-7)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{65}}{4} \end{aligned}$$

Step 5: Write the final answer

The two roots are $x = \frac{-3 + \sqrt{65}}{4}$ or $x = \frac{-3 - \sqrt{65}}{4}$.

Worked example 9: Using the quadratic formula

QUESTION

Find the roots of the function $f(x) = x^2 - 5x + 8$.

SOLUTION

Step 1: Finding the roots

To determine the roots of $f(x)$, we let $x^2 - 5x + 8 = 0$.

Step 2: Check whether the expression can be factorised

The expression cannot be factorised, so the general quadratic formula must be used.

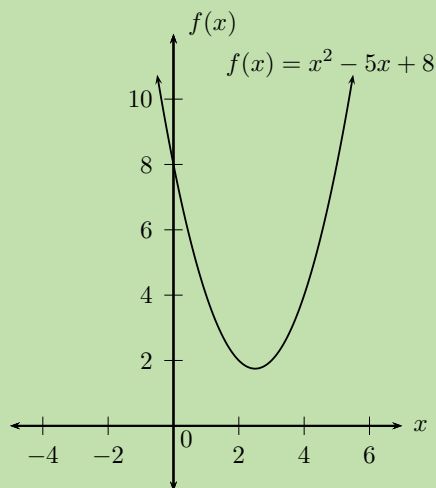
Step 3: Identify the coefficients to substitute into the formula

$$a = 1; \qquad b = -5; \qquad c = 8$$

Step 4: Apply the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)} \\ &= \frac{5 \pm \sqrt{-7}}{2} \end{aligned}$$

Step 5: Write the final answer

There are no real roots for $f(x) = x^2 - 5x + 8$ since the expression under the square root is negative ($\sqrt{-7}$ is not a real number). This means that the graph of the quadratic function has no x-intercepts; the entire graph lies above the x-axis.

See video: 2288 at www.everythingmaths.co.za

Exercise 2 – 3: Solution by the quadratic formula

Solve the following using the quadratic formula.

1. $3t^2 + t - 4 = 0$
2. $x^2 - 5x - 3 = 0$
3. $2t^2 + 6t + 5 = 0$
4. $2p(2p + 1) = 2$
5. $-3t^2 + 5t - 8 = 0$
6. $5t^2 + 3t - 3 = 0$
7. $t^2 - 4t + 2 = 0$
8. $9(k^2 - 1) = 7k$
9. $3f - 2 = -2f^2$
10. $t^2 + t + 1 = 0$

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1. 2289 2. 228B 3. 228C 4. 228D 5. 228F 6. 228G
7. 228H 8. 228J 9. 228K 10. 228M

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2.4 Substitution EMBFM

It is often useful to make a substitution for a repeated expression in a quadratic equation. This makes the equation simpler and much easier to solve.

Worked example 10: Solving by substitution

QUESTION

Solve for x: $x^2 - 2x - \frac{3}{x^2 - 2x} = 2$

SOLUTION

Step 1: Determine the restrictions for x

The restrictions are the values for x that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $x \neq 0$ and $x \neq 2$.

Step 2: Substitute a single variable for the repeated expression

We notice that $x^2 - 2x$ is a repeated expression and we therefore let $k = x^2 - 2x$ so that the equation becomes

$$k - \frac{3}{k} = 2$$

Step 3: Determine the restrictions for k

The restrictions are the values for k that would result in the denominator being equal to 0, which would make the fraction undefined. Therefore $k \neq 0$.

Step 4: Solve for k

$$k - \frac{3}{k} = 2$$

$$k^2 - 3 = 2k$$

$$k^2 - 2k - 3 = 0$$

$$(k + 1)(k - 3) = 0$$

$$\text{Therefore } k = -1 \text{ or } k = 3$$

We check these two roots against the restrictions for k and confirm that both are valid.

Step 5: Use values obtained for k to solve for the original variable x

For $k = -1$

$$x^2 - 2x = -1$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

$$\text{Therefore } x = 1$$

For $k = 3$

$$x^2 - 2x = 3$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$\text{Therefore } x = -1 \text{ or } x = 3$$

We check these roots against the restrictions for x and confirm that all three values are valid.

Step 6: Write the final answer

The roots of the equation are $x = -1$, $x = 1$ and $x = 3$.

Exercise 2 – 4:

Solve the following quadratic equations by substitution:

1. $-24 = 10(x^2 + 5x) + (x^2 + 5x)^2$
2. $(x^2 - 2x)^2 - 8 = 7(x^2 - 2x)$
3. $x^2 + 3x - \frac{56}{x(x+3)} = 26$
4. $x^2 - 18 + x + \frac{72}{x^2 + x} = 0$
5. $x^2 - 4x + 10 - 7(4x - x^2) = -2$
6. $\frac{9}{x^2 + 2x - 12} = x^2 + 2x - 12$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 228N 2. 228P 3. 228Q 4. 228R 5. 228S 6. 228T

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2.5 Finding the equation

EMBFN

We have seen that the roots are the solutions obtained from solving a quadratic equation. Given the roots, we are also able to work backwards to determine the original quadratic equation.

Worked example 11: Finding an equation when the roots are given

QUESTION

Find an equation with roots 13 and -5.

SOLUTION

Step 1: Assign a variable and write roots as two equations

$$x = 13 \text{ or } x = -5$$

Use additive inverses to get zero on the right-hand sides

$$x - 13 = 0 \text{ or } x + 5 = 0$$

Step 2: Write down as the product of two factors

$$(x - 13)(x + 5) = 0$$

Notice that the signs in the brackets are opposite of the given roots.

Step 3: Expand the brackets

$$x^2 - 8x - 65 = 0$$

Note that if each term in the equation is multiplied by a constant then there could be other possible equations which would have the same roots. For example,

Multiply by 2:

$$2x^2 - 16x - 130 = 0$$

Multiply by -3:

$$-3x^2 + 24x + 195 = 0$$

Worked example 12: Finding an equation when the roots are fractions

QUESTION

Find an equation with roots $-\frac{3}{2}$ and 4.

SOLUTION

Step 1: Assign a variable and write roots as two equations

$$x = 4 \text{ or } x = -\frac{3}{2}$$

Use additive inverses to get zero on the right-hand sides.

$$x - 4 = 0 \text{ or } x + \frac{3}{2} = 0$$

Multiply the second equation through by 2 to remove the fraction.

$$x - 4 = 0 \text{ or } 2x + 3 = 0$$

Step 2: Write down as the product of two factors

$$(2x + 3)(x - 4) = 0$$

Step 3: Expand the brackets

The quadratic equation is $2x^2 - 5x - 12 = 0$.

Exercise 2 – 5: Finding the equation

1. Determine a quadratic equation for a graph that has roots 3 and -2.
2. Find a quadratic equation for a graph that has x-intercepts of $(-4; 0)$ and $(4; 0)$.
3. Determine a quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c are integers, that has roots $-\frac{1}{2}$ and 3.
4. Determine the value of k and the other root of the quadratic equation $kx^2 - 7x + 4 = 0$ given that one of the roots is $x = 1$.
5. One root of the equation $2x^2 - 3x = p$ is $2\frac{1}{2}$. Find p and the other root.

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1. 228V 2. 228W 3. 228X 4. 228Y 5. 228Z

Exercise 2 – 6: Mixed exercises

Solve the following quadratic equations by either factorisation, using the quadratic formula or completing the square:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- In a test or examination, only use the method of completing the square when specifically asked.
- Answers can be left in surd or decimal form.

1. $24y^2 + 61y - 8 = 0$
2. $8x^2 + 16x = 42$
3. $9t^2 = 24t - 12$
4. $-5y^2 + 0y + 5 = 0$
5. $3m^2 + 12 = 15m$
6. $49y^2 + 0y - 25 = 0$
7. $72 = 66w - 12w^2$
8. $-40y^2 + 58y - 12 = 0$
9. $37n + 72 - 24n^2 = 0$
10. $6y^2 + 7y - 24 = 0$
11. $3 = x(2x - 5)$
12. $-18y^2 - 55y - 25 = 0$
13. $-25y^2 + 25y - 4 = 0$
14. $8(1 - 4g^2) + 24g = 0$
15. $9y^2 - 13y - 10 = 0$
16. $(7p - 3)(5p + 1) = 0$
17. $-81y^2 - 99y - 18 = 0$
18. $14y^2 - 81y + 81 = 0$

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1. 2292 2. 2293 3. 2294 4. 2295 5. 2296 6. 2297
7. 2298 8. 2299 9. 229B 10. 229C 11. 229D 12. 229F
13. 229G 14. 229H 15. 229J 16. 229K 17. 229M 18. 229N

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2.6 Nature of roots EMBFP

Investigation:

1. Use the quadratic formula to determine the roots of the quadratic equations given below and take special note of:
   - the expression under the square root sign and
   - the type of number for the final answer (rational/irrational/real/imaginary)

   a) $x^2 - 6x + 9 = 0$

b) $x^2 - 4x + 3 = 0$

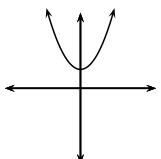
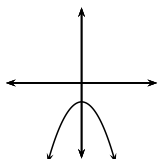
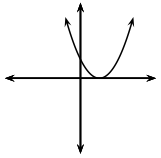
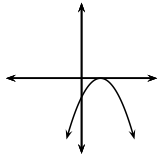
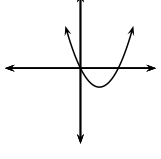
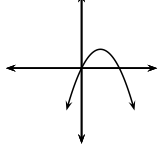
c) $x^2 - 4x - 3 = 0$

d) $x^2 - 4x + 7 = 0$

2. Choose the appropriate words from the table to describe the roots obtained for the equations above.

rational	unequal	real
imaginary	not perfect square	equal
perfect square	irrational	undefined

3. The expression under the square root, $b^2 - 4ac$, is called the discriminant. Can you make a conjecture about the relationship between the discriminant and the roots of quadratic equations?

The discriminant EMBFQ

The discriminant is defined as $\Delta = b^2 - 4ac$

This is the expression under the square root in the quadratic formula. The discriminant determines the nature of the roots of a quadratic equation. The word 'nature' refers to the types of numbers the roots can be — namely real, rational, irrational or imaginary. Δ is the Greek symbol for the letter D.

For a quadratic function $f(x) = ax^2 + bx + c$, the solutions to the equation $f(x) = 0$ are given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$

- If $\Delta < 0$, then roots are imaginary (non-real) and beyond the scope of this book.
- If $\Delta \geq 0$, the expression under the square root is non-negative and therefore roots are real. For real roots, we have the following further possibilities.
- If $\Delta = 0$, the roots are equal and we can say that there is only one root.
- If $\Delta > 0$, the roots are unequal and there are two further possibilities.
- Δ is the square of a rational number: the roots are rational.
- Δ is not the square of a rational number: the roots are irrational and can be expressed in decimal or surd form.

Nature of roots	Discriminant	$a > 0$	$a < 0$
Roots are non-real	$\Delta < 0$		
Roots are real and equal	$\Delta = 0$		
Roots are real and unequal: • rational roots • irrational roots	$\Delta > 0$ • $\Delta =$ squared rational • $\Delta =$ not squared rational		

See video: 229P at www.everythingmaths.co.za

Worked example 13: Nature of roots

QUESTION

Show that the roots of $x^2 - 2x - 7 = 0$ are irrational.

SOLUTION

Step 1: Interpret the question

For roots to be real and irrational, we need to calculate Δ and show that it is greater than zero and not a perfect square.

Step 2: Check that the equation is in standard form $ax^2 + bx + c = 0$

$$x^2 - 2x - 7 = 0$$

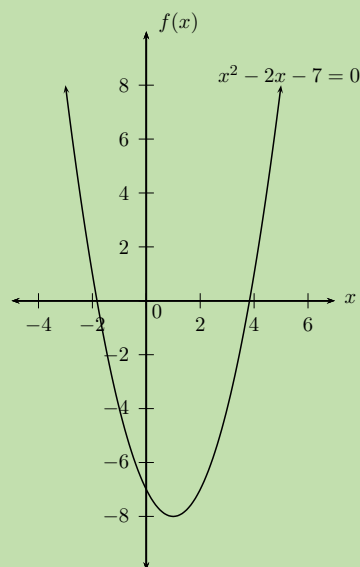
Step 3: Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = -2; \qquad c = -7$$

Step 4: Write down the formula and substitute values

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(-7) \\ &= 4 + 28 \\ &= 32 \end{aligned}$$

We know that $32 > 0$ and is not a perfect square.

The graph below shows the roots of the equation $x^2 - 2x - 7 = 0$. Note that the graph does not form part of the answer and is included for illustration purposes only.

Step 5: Write the final answer

We have calculated that $\Delta > 0$ and is not a perfect square, therefore we can conclude that the roots are **real, unequal** and **irrational**.

Worked example 14: Nature of roots

QUESTION

For which value(s) of k will the roots of $6x^2 + 6 = 4kx$ be real and equal?

SOLUTION

Step 1: Interpret the question

For roots to be real and equal, we need to solve for the value(s) of k such that $\Delta = 0$.

Step 2: Check that the equation is in standard form $ax^2 + bx + c = 0$

$$6x^2 - 4kx + 6 = 0$$

Step 3: Identify the coefficients to substitute into the formula for the discriminant

$$a = 6; \qquad b = -4k; \qquad c = 6$$

Step 4: Write down the formula and substitute values

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-4k)^2 - 4(6)(6) \\ &= 16k^2 - 144\end{aligned}$$

For roots to be real and equal, $\Delta = 0$.

$$\begin{aligned}\Delta &= 0 \\ 16k^2 - 144 &= 0 \\ 16(k^2 - 9) &= 0 \\ (k-3)(k+3) &= 0\end{aligned}$$

Therefore $k = 3$ or $k = -3$.

Step 5: Check both answers by substituting back into the original equation

For $k = 3$:

$$\begin{aligned} 6x^2 - 4(3)x + 6 &= 0 \\ 6x^2 - 12x + 6 &= 0 \\ x^2 - 2x + 1 &= 0 \\ (x-1)(x-1) &= 0 \\ (x-1)^2 &= 0 \\ \text{Therefore } x &= 1 \end{aligned}$$

We see that for $k = 3$ the quadratic equation has real, equal roots $x = 1$.

For $k = -3$:

$$\begin{aligned} 6x^2 - 4(-3)x + 6 &= 0 \\ 6x^2 + 12x + 6 &= 0 \\ x^2 + 2x + 1 &= 0 \\ (x+1)(x+1) &= 0 \\ (x+1)^2 &= 0 \\ \text{Therefore } x &= -1 \end{aligned}$$

We see that for $k = -3$ the quadratic equation has real, equal roots $x = -1$.

Step 6: Write the final answer

For the roots of the quadratic equation to be **real** and **equal**, $k = 3$ or $k = -3$.

Worked example 15: Nature of roots

QUESTION

Show that the roots of $(x + h)(x + k) = 4d^2$ are real for all real values of h, k and d.

SOLUTION

Step 1: Interpret the question

For roots to be real, we need to calculate Δ and show that $\Delta \geq 0$ for all real values of h, k and d.

Step 2: Check that the equation is in standard form $ax^2 + bx + c = 0$

Expand the brackets and gather like terms

$$
\begin{aligned}
(x+h)(x+k) &= 4d^2 \\
x^2 + hx + kx + hk - 4d^2 &= 0 \\
x^2 + (h+k)x + (hk - 4d^2) &= 0
\end{aligned}
$$

Step 3: Identify the coefficients to substitute into the formula for the discriminant

$$a = 1; \qquad b = h + k; \qquad c = hk - 4d^2$$

Step 4: Write down the formula and substitute values

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (h+k)^2 - 4(1)(hk - 4d^2) \\
&= h^2 + 2hk + k^2 - 4hk + 16d^2 \\
&= h^2 - 2hk + k^2 + 16d^2 \\
&= (h-k)^2 + (4d)^2
\end{aligned}
$$

For roots to be real, $\Delta \geq 0$.

$$
\begin{aligned}
\text{We know that } (4d)^2 &\geq 0 \\
\text{and } (h-k)^2 &\geq 0 \\
\text{so then } (h-k)^2 + (4d)^2 &\geq 0 \\
\text{therefore } \Delta &\geq 0
\end{aligned}
$$

Step 5: Write the final answer

We have shown that $\Delta \geq 0$, therefore the roots are **real** for all real values of h, k and d.

Exercise 2 – 7: From past papers

1. Determine the nature of the roots for each of the following equations:

   a) $x^2 + 3x = -2$

   b) $x^2 + 9 = 6x$

   c) $6y^2 - 6y - 1 = 0$

   d) $4t^2 - 19t - 5 = 0$

   e) $z^2 = 3$

   f) $0 = p^2 + 5p + 8$

   g) $x^2 = 36$

   h) $4m + m^2 = 1$

   i) $11 - 3x + x^2 = 0$

   j) $y^2 + \frac{1}{4} = y$

2. Given: $x^2 + bx - 2 + k\left(x^2 + 3x + 2\right) = 0$, $(k \neq -1)$

   a) Show that the discriminant is given by: $\Delta = k^2 + 6bk + b^2 + 8$

b) If $b = 0$, discuss the nature of the roots of the equation.

c) If $b = 2$, find the value(s) of k for which the roots are equal.

[IEB, Nov. 2001, HG]

3. Show that $k^2x^2 + 2 = kx - x^2$ has non-real roots for all real values for k.

   [IEB, Nov. 2002, HG]

4. The equation $x^2 + 12x = 3kx^2 + 2$ has real roots.

   a) Find the greatest value of value k such that $k \in \mathbb{Z}$.

   b) Find one rational value of k for which the above equation has rational roots.

   [IEB, Nov. 2003, HG]

5. Consider the equation:

$$k = \frac{x^2 - 4}{2x - 5}$$

   where $x \neq \frac{5}{2}$.

   a) Find a value of k for which the roots are equal.

   b) Find an integer k for which the roots of the equation will be rational and unequal.

   [IEB, Nov. 2004, HG]

6. a) Prove that the roots of the equation $x^2 - (a + b)x + ab - p^2 = 0$ are real for all real values of a, b and p.

   b) When will the roots of the equation be equal?

   [IEB, Nov. 2005, HG]

7. If b and c can take on only the values 1, 2 or 3, determine all pairs $(b; c)$ such that $x^2 + bx + c = 0$ has real roots.

   [IEB, Nov. 2005, HG]

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1a. 229Q 1b. 229R 1c. 229S 1d. 229T 1e. 229V 1f. 229W
1g. 229X 1h. 229Y 1i. 229Z 1j. 22B2 2. 22B3 3. 22B4
4. 22B5 5. 22B6 6. 22B7 7. 22B8

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2.7 Quadratic inequalities EMBFR

Quadratic inequalities can be of the following forms:

$$ax^2 + bx + c > 0$$
$$ax^2 + bx + c \geq 0$$
$$ax^2 + bx + c < 0$$
$$ax^2 + bx + c \leq 0$$

To solve a quadratic inequality we must determine which part of the graph of a quadratic function lies above or below the x-axis. An inequality can therefore be solved graphically using a graph or algebraically using a table of signs to determine where the function is positive and negative.

Worked example 16: Solving quadratic inequalities

QUESTION

Solve for x: $x^2 - 5x + 6 \geq 0$

SOLUTION

Step 1: Factorise the quadratic

$$(x - 3)(x - 2) \geq 0$$

Step 2: Determine the critical values of x

From the factorised quadratic we see that the values for which the inequality is equal to zero are $x = 3$ and $x = 2$. These are called the critical values of the inequality and they are used to complete a table of signs.

Step 3: Complete a table of signs

We must determine where each factor of the inequality is positive and negative on the number line:

- to the left (in the negative direction) of the critical value
- equal to the critical value
- to the right (in the positive direction) of the critical value

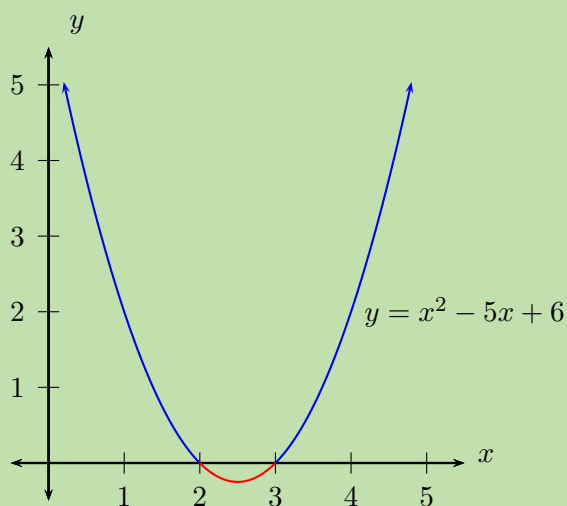
In the final row of the table we determine where the inequality is positive and negative by finding the product of the factors and their respective signs.

Critical values		$x = 2$		$x = 3$	
$x - 3$	$-$	$-$	$-$	0	$+$
$x - 2$	$-$	0	$+$	$+$	$+$
$f(x) = (x - 3)(x - 2)$	$+$	0	$-$	0	$+$

From the table we see that $f(x)$ is greater than or equal to zero for $x \leq 2$ or $x \geq 3$.

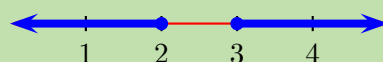
Step 4: A rough sketch of the graph

The graph below does not form part of the answer and is included for illustration purposes only. A graph of the quadratic helps us determine the answer to the inequality. We can find the answer graphically by seeing where the graph lies above or below the x-axis.

- From the standard form, $x^2 - 5x + 6$, $a > 0$ and therefore the graph is a "smile" and has a minimum turning point.
- From the factorised form, $(x - 3)(x - 2)$, we know the x-intercepts are $(2; 0)$ and $(3; 0)$.

The graph is above or on the x-axis for $x \leq 2$ or $x \geq 3$.

Step 5: Write the final answer and represent on a number line

$$x^2 - 5x + 6 \geq 0 \text{ for } x \leq 2 \text{ or } x \geq 3$$

1 2 3 4

Worked example 17: Solving quadratic inequalities

QUESTION

Solve for x: $4x^2 - 4x + 1 \leq 0$

SOLUTION

Step 1: Factorise the quadratic

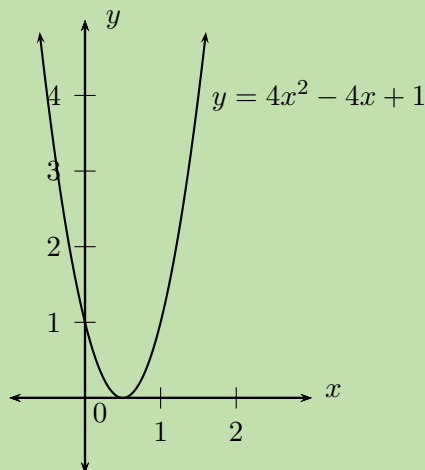
$$(2x - 1)(2x - 1) \leq 0$$
$$(2x - 1)^2 \leq 0$$

Step 2: Determine the critical values of x

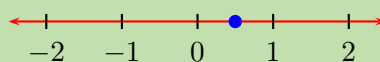
From the factorised quadratic we see that the value for which the inequality is equal to zero is $x = \frac{1}{2}$. We know that $a^2 > 0$ for any real number $a, a \neq 0$, so then $(2x - 1)^2$ will never be negative.

Step 3: A rough sketch of the graph

The graph below does not form part of the answer and is included for illustration purposes only.

- From the standard form, $4x^2 - 4x + 1$, $a > 0$ and therefore the graph is a "smile" and has a minimum turning point.
- From the factorised form, $(2x-1)(2x-1)$, we know there is only one x-intercept at $\left(\frac{1}{2}; 0\right)$.

Notice that no part of the graph lies below the x-axis.

Step 4: Write the final answer and represent on a number line

$$4x^2 - 4x + 1 \leq 0 \text{ for } x = \frac{1}{2}$$

Worked example 18: Solving quadratic inequalities

QUESTION

Solve for x: $-x^2 - 3x + 5 > 0$

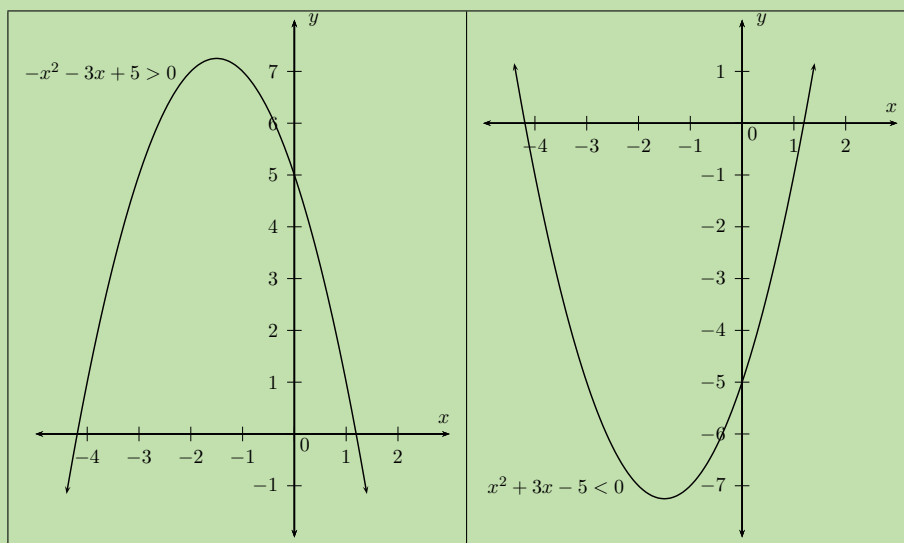
SOLUTION

Step 1: Examine the form of the inequality

Notice that the coefficient of the x^2 term is -1. **Remember** that if we multiply or divide an inequality by a negative number, then the inequality sign changes direction. So we can write the same inequality in different ways and still get the same answer, as shown below.

$$-x^2 - 3x + 5 > 0$$

Multiply by -1 and change direction of the inequality sign

$$x^2 + 3x - 5 < 0$$

From this rough sketch, we can see that both inequalities give the same solution; the values of x that lie between the two x-intercepts.

Step 2: Factorise the quadratic

We notice that $-x^2-3x+5 > 0$ cannot be easily factorised. So we let $-x^2-3x+5 = 0$ and use the quadratic formula to determine the roots of the equation.

$$-x^2 - 3x + 5 = 0$$
$$x^2 + 3x - 5 = 0$$

$$
\begin{aligned}
\therefore x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{29}}{2} \\
x_1 &= \frac{-3 - \sqrt{29}}{2} \approx -4{,}2 \\
x_2 &= \frac{-3 + \sqrt{29}}{2} \approx 1{,}2
\end{aligned}
$$

Therefore we can write, correct to one decimal place,

$$
\begin{aligned}
x^2 + 3x - 5 &< 0 \\
\text{as } (x - 1{,}2)(x + 4{,}2) &< 0
\end{aligned}
$$

Step 3: Determine the critical values of x

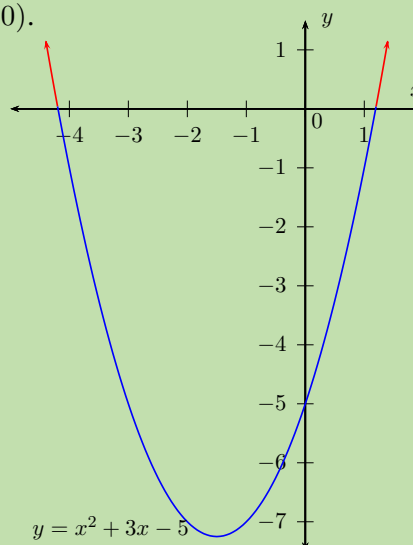
From the factorised quadratic we see that the critical values are $x = 1{,}2$ and $x = -4{,}2$.

Step 4: Complete a table of signs

Critical values		$x = -4{,}2$		$x = 1{,}2$	
$x + 4{,}2$	$-$	0	$+$	$+$	$+$
$x - 1{,}2$	$-$	$-$	$-$	0	$+$
$f(x) = (x + 4{,}2)(x - 1{,}2)$	$+$	0	$-$	0	$+$

From the table we see that the function is negative for $-4{,}2 < x < 1{,}2$.

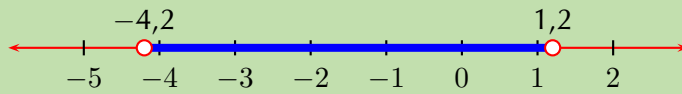
Step 5: A sketch of the graph

- From the standard form, $x^2 + 3x - 5$, $a > 0$ and therefore the graph is a "smile" and has a minimum turning point.
- From the factorised form, $(x - 1{,}2)(x + 4{,}2)$, we know the x-intercepts are $(-4{,}2; 0)$ and $(1{,}2; 0)$.

From the graph we see that the function lies below the x-axis between $-4{,}2$ and $1{,}2$.

Step 6: Write the final answer and represent on a number line

$$x^2 + 3x - 5 < 0 \text{ for } -4{,}2 < x < 1{,}2$$

−4,2 1,2

−5 −4 −3 −2 −1 0 1 2

Important: When working with an inequality in which the variable is in the denominator, a different approach is needed. Always remember to check for restrictions.

Worked example 19: Solving quadratic inequalities with fractions

QUESTION

Solve for x:

1. $\frac{2}{x+3} = \frac{1}{x-3}, x \neq \pm 3$
2. $\frac{2}{x+3} \leq \frac{1}{x-3}, x \neq \pm 3$

SOLUTION

Step 1: Solving the equation

To solve this equation we multiply both sides of the equation by $(x+3)(x-3)$ and simplfy:

$$\frac{2}{x+3} \times (x+3)(x-3) = \frac{1}{x-3} \times (x+3)(x-3)$$

$$2(x-3) = x+3$$

$$2x - 6 = x + 3$$

$$x = 9$$

Step 2: Solving the inequality

It is very important to recognise that we cannot use the same method as above to solve the inequality. If we multiply or divide an inequality by a negative number, then the inequality sign changes direction. We must rather simplify the inequality to have a lowest common denominator and use a table of signs to determine the values that satisfy the inequality.

Step 3: Subtract $\frac{1}{x-3}$ from both sides of the inequality

$$\frac{2}{x+3} - \frac{1}{x-3} \leq 0$$

Step 4: Determine the lowest common denominator and simplify the fraction

$$\frac{2(x-3)-(x+3)}{(x+3)(x-3)} \leq 0$$
$$\frac{x-9}{(x+3)(x-3)} \leq 0$$

Keep the denominator because it affects the final answer.

Step 5: Determine the critical values of x

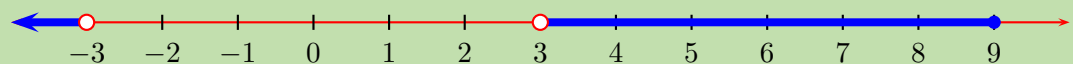
From the factorised inequality we see that the critical values are $x = -3$, $x = 3$ and $x = 9$.

Step 6: Complete a table of signs

Critical values		$x = -3$		$x = 3$		$x = 9$	
$x+3$	$-$	undef	$+$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	undef	$+$	$+$	$+$
$x-9$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x) = \frac{x-9}{(x+3)(x-3)}$	$-$	undef	$+$	undef	$-$	0	$+$

From the table we see that the function is less than or equal to zero for $x < -3$ or $3 < x \leq 9$. We do not include $x = -3$ or $x = 3$ in the solution because of the restrictions on the denominator.

Step 7: Write the final answer and represent on a number line

$$x < -3 \quad \text{or} \quad 3 < x \leq 9$$

Exercise 2 – 8: Solving quadratic inequalities

1. Solve the following inequalities and show each answer on a number line:

a) $x^2 - x < 12$

b) $3x^2 > -x + 4$

c) $y^2 < -y - 2$

d) $(3 - t)(1 + t) > 0$

e) $s^2 - 4s > -6$

f) $0 \geq 7x^2 - x + 8$

g) $x \geq -4x^2$

h) $2x^2 + x + 6 \leq 0$

i) $\frac{x}{x-3} < 2, x \neq 3$

j) $\frac{x^2+4}{x-7} \geq 0, x \neq 7$

k) $\frac{x+2}{x} - 1 \geq 0, x \neq 0$

2. Draw a sketch of the following inequalities and solve for x:

a) $2x^2 - 18 > 0$

b) $5 - x^2 \leq 0$

c) $x^2 < 0$

d) $0 \geq 6x^2$

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1a. 22B9 1b. 22BB 1c. 22BC 1d. 22BD 1e. 22BF 1f. 22BG
1g. 22BH 1h. 22BJ 1i. 22BK 1j. 22BM 1k. 22BN 2a. 22BP
2b. 22BQ 2c. 22BR 2d. 22BS

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2.8 Simultaneous equations EMBFS

Simultaneous linear equations can be solved using three different methods: substitution, elimination or using a graph to determine where the two lines intersect. For solving systems of simultaneous equations with linear and non-linear equations, we mostly use the substitution method. Graphical solution is useful for showing where the two equations intersect.

In general, to solve for the values of n unknown variables requires a system of n independent equations.

An example of a system of simultaneous equations with one linear equation and one quadratic equation is

$$y - 2x = -4$$
$$x^2 + y = 4$$

Solving by substitution EMBFT

- Use the simplest of the two given equations to express one of the variables in terms of the other.
- Substitute into the second equation. By doing this we reduce the number of equations and the number of variables by one.

- We now have one equation with one unknown variable which can be solved.
- Use the solution to substitute back into the first equation to find the value of the other unknown variable.

Worked example 20: Simultaneous equations

QUESTION

Solve for x and y:

$$\begin{aligned} y - 2x &= -4 \qquad \ldots(1) \\ x^2 + y &= 4 \qquad \ldots(2) \end{aligned}$$

SOLUTION

Step 1: Make y the subject of the first equation

$$y = 2x - 4$$

Step 2: Substitute into the second equation and simplify

$$\begin{aligned} x^2 + (2x - 4) &= 4 \\ x^2 + 2x - 8 &= 0 \end{aligned}$$

Step 3: Factorise the equation

$$\begin{aligned} (x + 4)(x - 2) &= 0 \\ \therefore x &= -4 \text{ or } x = 2 \end{aligned}$$

Step 4: Substitute the values of x back into the first equation to determine the corresponding y-values

If $x = -4$:

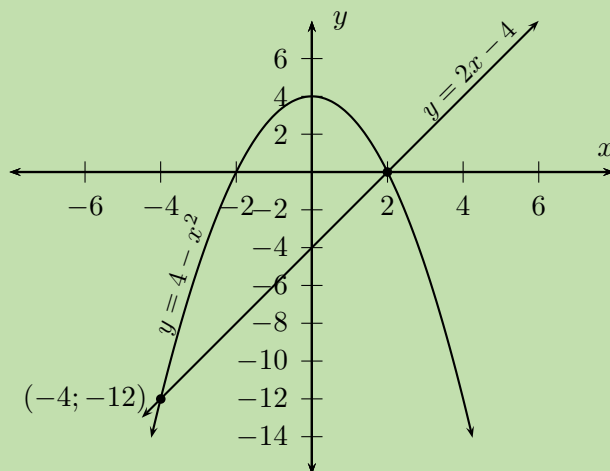
$$\begin{aligned} y &= 2(-4) - 4 \\ &= -12 \end{aligned}$$

If $x = 2$:

$$\begin{aligned} y &= 2(2) - 4 \\ &= 0 \end{aligned}$$

Step 5: Check that the two points satisfy both original equations

Step 6: Write the final answer

The solution is $x = -4$ and $y = -12$ or $x = 2$ and $y = 0$. These are the coordinate pairs for the points of intersection as shown below.

Solving by elimination

EMBFV

- Make one of the variables the subject of both equations.
- Equate the two equations; by doing this we reduce the number of equations and the number of variables by one.
- We now have one equation with one unknown variable which can be solved.
- Use the solution to substitute back into either original equation, to find the corresponding value of the other unknown variable.

Worked example 21: Simultaneous equations

QUESTION

Solve for x and y:

$$y = x^2 - 6x \qquad \ldots(1)$$
$$y + \tfrac{1}{2}x - 3 = 0 \qquad \ldots(2)$$

SOLUTION

Step 1: Make y the subject of the second equation

$$y + \frac{1}{2}x - 3 = 0$$
$$y = -\frac{1}{2}x + 3$$

Step 2: Equate the two equations and solve for x

$$\begin{aligned} x^2 - 6x &= -\frac{1}{2}x + 3 \\ x^2 - 6x + \frac{1}{2}x - 3 &= 0 \\ 2x^2 - 12x + x - 6 &= 0 \\ 2x^2 - 11x - 6 &= 0 \\ (2x + 1)(x - 6) &= 0 \\ \text{Therefore } x = -\frac{1}{2} &\text{ or } x = 6 \end{aligned}$$

Step 3: Substitute the values for x back into the second equation to calculate the corresponding y-values

If $x = -\frac{1}{2}$:

$$\begin{aligned} y &= -\frac{1}{2}\left(-\frac{1}{2}\right) + 3 \\ \therefore y &= 3\frac{1}{4} \end{aligned}$$

This gives the point $\left(-\frac{1}{2}; 3\frac{1}{4}\right)$.

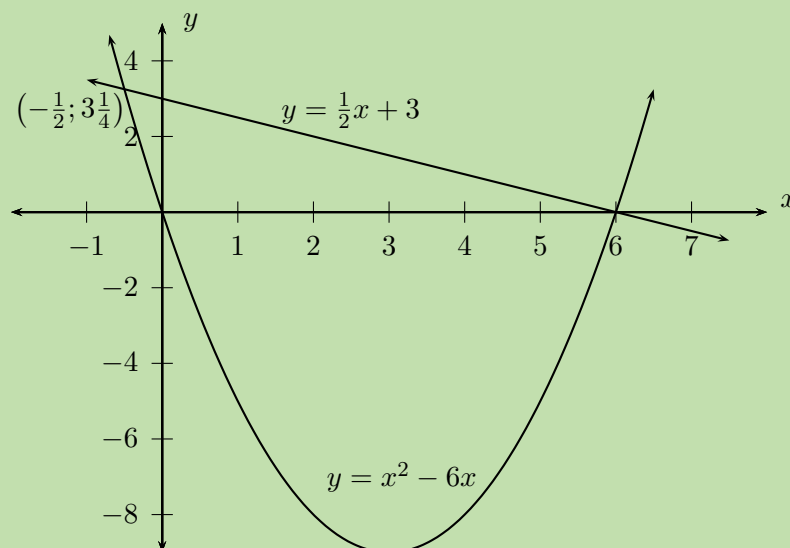
If $x = 6$:

$$\begin{aligned} y &= -\frac{1}{2}(6) + 3 \\ &= -3 + 3 \\ \therefore y &= 0 \end{aligned}$$

This gives the point $(6; 0)$.

Step 4: Check that the two points satisfy both original equations

Step 5: Write the final answer

The solution is $x = -\frac{1}{2}$ and $y = 3\frac{1}{4}$ or $x = 6$ and $y = 0$. These are the coordinate pairs for the points of intersection as shown below.

Worked example 22: Simultaneous equations

QUESTION

Solve for x and y:

$$y = \frac{5}{x-2} \qquad \ldots(1)$$
$$y + 1 = 2x \qquad \ldots(2)$$

SOLUTION

Step 1: Make y the subject of the second equation

$$\begin{aligned} y + 1 &= 2x \\ y &= 2x - 1 \end{aligned}$$

Step 2: Equate the two equations and solve for x

$$\begin{aligned} 2x - 1 &= \frac{5}{x-2} \\ (2x-1)(x-2) &= 5 \\ 2x^2 - 5x + 2 &= 5 \\ 2x^2 - 5x - 3 &= 0 \\ (2x+1)(x-3) &= 0 \\ \text{Therefore } x = -\frac{1}{2} &\text{ or } x = 3 \end{aligned}$$

Step 3: Substitute the values for x back into the second equation to calculate the corresponding y-values

If $x - \frac{1}{2}$:

$$\begin{aligned} y &= 2(-\frac{1}{2}) - 1 \\ \therefore y &= -2 \end{aligned}$$

This gives the point $(-\frac{1}{2}; -2)$.

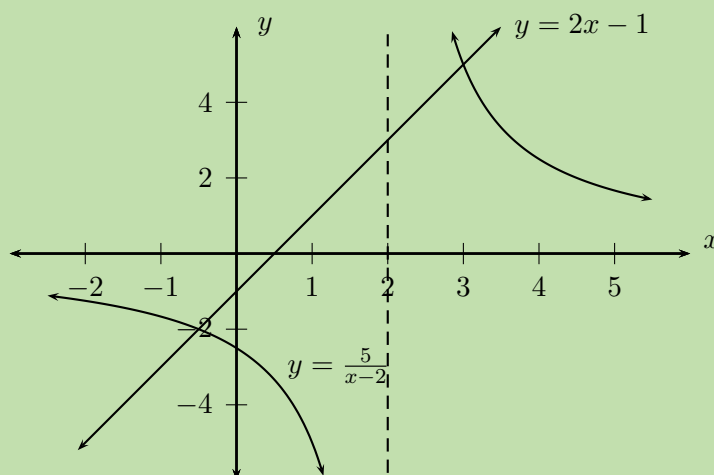
If $x = 3$:

$$\begin{aligned} y &= 2(3) - 1 \\ &= 5 \end{aligned}$$

This gives the point $(3; 5)$.

Step 4: Check that the two points satisfy both original equations

Step 5: Write the final answer

The solution is $x = -\frac{1}{2}$ and $y = -2$ or $x = 3$ and $y = 5$. These are the coordinate pairs for the points of intersection as shown below.

Solving graphically

EMBFW

- Make y the subject of each equation.
- Draw the graph of each equation on the same system of axes.
- The final solutions to the system of equations are the coordinates of the points where the two graphs intersect.

Worked example 23: Simultaneous equations

QUESTION

Solve graphically for x and y:

$$y + x^2 = 1 \qquad \ldots(1)$$
$$y - x + 5 = 0 \qquad \ldots(2)$$

SOLUTION

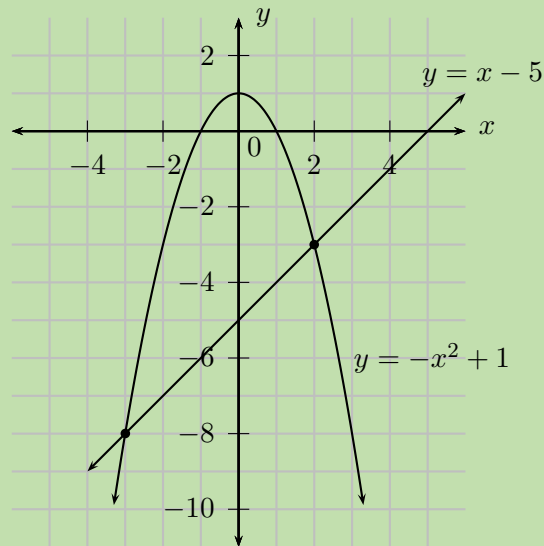
Step 1: Make y the subject of both equations

For the first equation we have

$$y + x^2 = 1$$
$$y = -x^2 + 1$$

and for the second equation

$$y - x + 5 = 0$$
$$y = x - 5$$

Step 2: Draw the straight line graph and parabola on the same system of axes

Step 3: Determine where the two graphs intersect

From the diagram we see that the graphs intersect at $(-3; -8)$ and $(2; -3)$.

Step 4: Check that the two points satisfy both original equations

Step 5: Write the final answer

The solutions to the system of simultaneous equations are $(-3; -8)$ and $(2; -3)$.

Exercise 2 – 9: Solving simultaneous equations

1. Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

a) $y + x = 5$
$y - x^2 + 3x - 5 = 0$

b) $y = 6 - 5x + x^2$
$y - x + 1 = 0$

c) $y = \frac{2x + 2}{4}$
$y - 2x^2 + 3x + 5 = 0$

d) $a - 2b - 3 = 0$; $a - 3b^2 + 4 = 0$

e) $x^2 - y + 2 = 3x$
$4x = 8 + y$

f) $2y + x^2 + 25 = 7x$
$3x = 6y + 96$

2. Solve the following systems of equations graphically. Check your solutions by also solving algebraically.

   a) $x^2 - 1 - y = 0$
      $y + x - 5 = 0$
   b) $x + y - 10 = 0$
      $x^2 - 2 - y = 0$
   c) $xy = 12$
      $7 = x + y$
   d) $6 - 4x - y = 0$
      $12 - 2x^2 - y = 0$

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1a. 22BT 1b. 22BV 1c. 22BW 1d. 22BX 1e. 22BY 1f. 22BZ
2a. 22C2 2b. 22C3 2c. 22C4 2d. 22C5

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2.9 Word problems EMBFX

Solving word problems requires using mathematical language to describe real-life contexts. Problem-solving strategies are often used in the natural sciences and engineering disciplines (such as physics, biology, and electrical engineering) but also in the social sciences (such as economics, sociology and political science). To solve word problems we need to write a set of equations that describes the problem mathematically.

Examples of real-world problem solving applications are:

- modelling population growth;
- modelling effects of air pollution;
- modelling effects of global warming;
- computer games;
- in the sciences, to understand how the natural world works;
- simulators that are used to train people in certain jobs, such as pilots, doctors and soldiers;
- in medicine, to track the progress of a disease.

See video: 22C6 at www.everythingmaths.co.za

Problem solving strategy

EMBFY

1. Read the problem carefully.
2. What is the question and what do we need to solve for?
3. Assign variables to the unknown quantities, for example, x and y.
4. Translate the words into algebraic expressions by rewriting the given information in terms of the variables.
5. Set up a system of equations.
6. Solve for the variables using substitution.
7. Check the solution.
8. Write the final answer.

Investigation: Simple word problems

Write an equation that describes the following real-world situations mathematically:

1. Mohato and Lindiwe both have colds. Mohato sneezes twice for each sneeze of Lindiwe's. If Lindiwe sneezes x times, write an equation describing how many times they both sneezed.
2. The difference of two numbers is 10 and the sum of their squares is 50. Find the two numbers.
3. Liboko builds a rectangular storeroom. If the diagonal of the room is $\sqrt{1312}$ m and the perimeter is 80 m, determine the dimensions of the room.
4. It rains half as much in July as it does in December. If it rains y mm in July, write an expression relating the rainfall in July and December.
5. Zane can paint a room in 4 hours. Tlali can paint a room in 2 hours. How long will it take both of them to paint a room together?
6. 25 years ago, Arthur was 5 years more than a third of Bongani's age. Today, Bongani is 26 years less than twice Arthur's age. How old is Bongani?
7. The product of two integers is 95. Find the integers if their total is 24.

Worked example 24: Gym membership

QUESTION

The annual gym subscription for a single member is R 1000, while an annual family membership is R 1500. The gym is considering increasing all membership fees by the same amount. If this is done then a single membership would cost $\frac{5}{7}$ of a family membership. Determine the amount of the proposed increase.

SOLUTION

Step 1: Identify the unknown quantity and assign a variable

Let the amount of the proposed increase be x.

Step 2: Use the given information to complete a table

	now	after increase
single	1000	$1000 + x$
family	1500	$1500 + x$

Step 3: Set up an equation

$$1000 + x = \frac{5}{7}(1500 + x)$$

Step 4: Solve for x

$$\begin{aligned} 7000 + 7x &= 7500 + 5x \\ 2x &= 500 \\ x &= 250 \end{aligned}$$

Step 5: Write the final answer

The proposed increase is R 250.

Worked example 25: Corner coffee house

QUESTION

Erica has decided to treat her friends to coffee at the Corner Coffee House. Erica paid R 54,00 for four cups of cappuccino and three cups of filter coffee. If a cup of cappuccino costs R 3,00 more than a cup of filter coffee, calculate how much a cup of each type of coffee costs?

SOLUTION

Step 1: Method 1: identify the unknown quantities and assign two variables

Let the cost of a cappuccino be x and the cost of a filter coffee be y.

Step 2: Use the given information to set up a system of equations

$$4x + 3y = 54 \quad \ldots (1)$$
$$x = y + 3 \quad \ldots (2)$$

Step 3: Solve the equations by substituting the second equation into the first equation

$$\begin{aligned} 4(y+3) + 3y &= 54 \\ 4y + 12 + 3y &= 54 \\ 7y &= 42 \\ y &= 6 \end{aligned}$$

If $y = 6$, then using the second equation we have

$$\begin{aligned} x &= y + 3 \\ &= 6 + 3 \\ &= 9 \end{aligned}$$

Step 4: Check that the solution satisfies both original equations

Step 5: Write the final answer

A cup of cappuccino costs R 9 and a cup of filter coffee costs R 6.

Step 6: Method 2: identify the unknown quantities and assign one variable

Let the cost of a cappuccino be x and the cost of a filter coffee be $x - 3$.

Step 7: Use the given information to set up an equation

$$4x + 3(x - 3) = 54$$

Step 8: Solve for x

$$\begin{aligned} 4x + 3(x-3) &= 54 \\ 4x + 3x - 9 &= 54 \\ 7x &= 63 \\ x &= 9 \end{aligned}$$

Step 9: Write the final answer

A cup of cappuccino costs R 9 and a cup of filter coffee costs R 6.

Worked example 26: Taps filling a container

QUESTION

Two taps, one more powerful than the other, are used to fill a container. Working on its own, the less powerful tap takes 2 hours longer than the other tap to fill the container. If both taps are opened, it takes 1 hour, 52 minutes and 30 seconds to fill the container. Determine how long it takes the less powerful tap to fill the container on its own.

SOLUTION

Step 1: Identify the unknown quantities and assign variables

Let the time taken for the less powerful tap to fill the container be x and let the time taken for the more powerful tap be $x - 2$.

Step 2: Convert all units of time to be the same

First we must convert 1 hour, 52 minutes and 30 seconds to hours:

$$1 + \frac{52}{60} + \frac{30}{(60)^2} = 1{,}875 \text{ hours}$$

Step 3: Use the given information to set up a system of equations

Write an equation describing the two taps working together to fill the container:

$$\frac{1}{x} + \frac{1}{x-2} = \frac{1}{1{,}875}$$

Step 4: Multiply the equation through by the lowest common denominator and simplify

$$\begin{aligned} 1{,}875(x-2) + 1{,}875x &= x(x-2) \\ 1{,}875x - 3{,}75 + 1{,}875x &= x^2 - 2x \\ 0 &= x^2 - 5{,}75x + 3{,}75 \end{aligned}$$

Multiply the equation through by 4 to make it easier to factorise (or use the quadratic formula)

$$\begin{aligned} 0 &= 4x^2 - 23x + 15 \\ 0 &= (4x-3)(x-5) \end{aligned}$$

Therefore $x = \frac{3}{4}$ or $x = 5$.

We have calculated that the less powerful tap takes $\frac{3}{4}$ hours or 5 hours to fill the container, but we know that when both taps are opened it takes 1,875 hours. We can therefore discard the first solution $x = \frac{3}{4}$ hours.

So the less powerful tap fills the container in 5 hours and the more powerful tap takes 3 hours.

Step 5: Check that the solution satisfies the original equation

Step 6: Write the final answer

The less powerful tap fills the container in 5 hours and the more powerful tap takes 3 hours.

Exercise 2 – 10:

1. Mr. Tsilatsila builds a fence around his rectangular vegetable garden of 8 m^2. If the length is twice the breadth, determine the dimensions of Mr. Tsilatsila's vegetable garden.
2. Kevin has played a few games of ten-pin bowling. In the third game, Kevin scored 80 more than in the second game. In the first game Kevin scored 110 less than the third game. His total score for the first two games was 208. If he wants an average score of 146, what must he score on the fourth game?
3. When an object is dropped or thrown downward, the distance, d, that it falls in time, t, is described by the following equation:

   $s = 5t^2 + v_0 t$

   In this equation, v_0 is the initial velocity, in $\text{m}\cdot\text{s}^{-1}$. Distance is measured in meters and time is measured in seconds. Use the equation to find how long it takes a tennis ball to reach the ground if it is thrown downward from a hot-air balloon that is 500 m high. The tennis ball is thrown at an initial velocity of 5 $\text{m}\cdot\text{s}^{-1}$.
4. The table below lists the times that Sheila takes to walk the given distances.

time (minutes)	5	10	15	20	25	30
distance (km)	1	2	3	4	5	6

   Plot the points.

   Find the equation that describes the relationship between time and distance. Then use the equation to answer the following questions:

   a) How long will it take Sheila to walk 21 km?
   b) How far will Sheila walk in 7 minutes?

   If Sheila were to walk half as fast as she is currently walking, what would the graph of her distances and times look like?
5. The power P (in watts) supplied to a circuit by a 12 volt battery is given by the formula $P = 12I - 0{,}5I^2$ where I is the current in amperes.

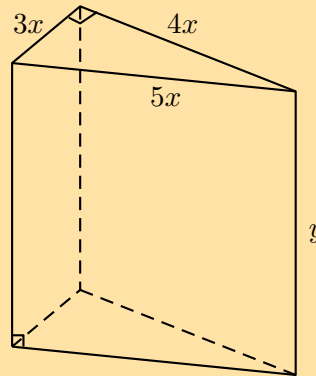
   a) Since both power and current must be greater than 0, find the limits of the current that can be drawn by the circuit.

b) Draw a graph of $P = 12I - 0{,}5I^2$ and use your answer to the first question to define the extent of the graph.

c) What is the maximum current that can be drawn?

d) From your graph, read off how much power is supplied to the circuit when the current is 10 A. Use the equation to confirm your answer.

e) At what value of current will the power supplied be a maximum?

6. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is 3600 cm^2.

Show that

$y = \frac{300 - x^2}{x}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22C7 2. 22C8 3. 22C9 4. 22CB 5. 22CC 6. 22CD

2.10 Summary

EMBFZ

See presentation: 22CF at www.everythingmaths.co.za

- Zero product law: if $a \times b = 0$, then $a = 0$ and/or $b = 0$.
- Quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Discriminant: $\Delta = b^2 - 4ac$

Nature of roots	Discriminant
Roots are non-real	$\Delta < 0$
Roots are real and equal	$\Delta = 0$
Roots are real and unequal: – Rational roots – Irrational roots	$\Delta > 0$ – $\Delta =$ squared rational number – $\Delta =$ not squared rational number

Exercise 2 – 11: End of chapter exercises

1. Solve: $x^2 - x - 1 = 0$. Give your answer correct to two decimal places.
2. Solve: $16(x+1) = x^2(x+1)$
3. Solve: $y^2 + 3 + \frac{12}{y^2+3} = 7$
4. Solve for x: $2x^4 - 5x^2 - 12 = 0$
5. Solve for x:
   a) $x(x-9) + 14 = 0$
   b) $x^2 - x = 3$ (correct to one decimal place)
   c) $x + 2 = \frac{6}{x}$ (correct to two decimal places)
   d) $\frac{1}{x+1} + \frac{2x}{x-1} = 1$
6. Solve for x in terms of p by completing the square: $x^2 - px - 4 = 0$
7. The equation $ax^2 + bx + c = 0$ has roots $x = \frac{2}{3}$ and $x = -4$. Find one set of possible values for a, b and c.
8. The two roots of the equation $4x^2 + px - 9 = 0$ differ by 5. Calculate the value of p.
9. An equation of the form $x^2 + bx + c = 0$ is written on the board. Saskia and Sven copy it down incorrectly. Saskia has a mistake in the constant term and obtains the solutions -4 and 2. Sven has a mistake in the coefficient of x and obtains the solutions 1 and -15. Determine the correct equation that was on the board.
10. For which values of b will the expression $\frac{b^2 - 5b + 6}{b+2}$ be:
    a) undefined?
    b) equal to zero?
11. Given $\frac{(x^2-6)(2x+1)}{x+2} = 0$ solve for x if:
    a) x is a real number.
    b) x is a rational number.
    c) x is an irrational number.
    d) x is an integer.

12. Given $\frac{(x-6)^{\frac{1}{2}}}{x^2+3}$, for which value(s) of x will the expression be:

a) equal to zero?

b) defined?

13. Solve for a if $\frac{\sqrt{8-2a}}{a-3} \geq 0$.

14. Abdoul stumbled across the following formula to solve the quadratic equation $ax^2 + bx + c = 0$ in a foreign textbook.

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

a) Use this formula to solve the equation: $2x^2 + x - 3 = 0$.

b) Solve the equation again, using factorisation, to see if the formula works for this equation.

c) Trying to derive this formula to prove that it always works, Abdoul got stuck along the way. His attempt is shown below:

$ax^2 + bx + c = 0$	
$a + \frac{b}{x} + \frac{c}{x^2} = 0$	Divided by x^2 where $x \neq 0$
$\frac{c}{x^2} + \frac{b}{x} + a = 0$	Rearranged
$\frac{1}{x^2} + \frac{b}{cx} + \frac{a}{c} = 0$	Divided by c where $c \neq 0$
$\frac{1}{x^2} + \frac{b}{cx} = -\frac{a}{c}$	Subtracted $\frac{a}{c}$ from both sides
$\therefore \frac{1}{x^2} + \frac{b}{cx} + \ldots$	Got stuck

Complete his derivation.

15. Solve for x:

a) $\frac{4}{x-3} \leq 1$

b) $\frac{4}{(x-3)^2} < 1$

c) $\frac{2x-2}{x-3} > 3$

d) $\frac{-3}{(x-3)(x+1)} < 0$

e) $(2x-3)^2 < 4$

f) $2x \leq \frac{15-x}{x}$

g) $\frac{x^2+3}{3x-2} \leq 0$

h) $x - 2 \geq \frac{3}{x}$

i) $\frac{x^2+3x-4}{5+x^4} \leq 0$

j) $\frac{x-2}{3-x} \geq 1$

16. Solve the following systems of equations algebraically. Leave your answer in surd form, where appropriate.

a) $y - 2x = 0$
$y - x^2 - 2x + 3 = 0$

b) $a - 3b = 0$
$a - b^2 + 4 = 0$

c) $y - x^2 - 5x = 0$
$10 = y - 2x$

d) $p = 2p^2 + q - 3$
$p - 3q = 1$

e) $a - b^2 = 0$
$a - 3b + 1 = 0$

f) $a - 2b + 1 = 0$
$a - 2b^2 - 12b + 4 = 0$

g) $y + 4x - 19 = 0$
$8y + 5x^2 - 101 = 0$

h) $a + 4b - 18 = 0$
$2a + 5b^2 - 57 = 0$

17. Solve the following systems of equations graphically:

a) $2y + x - 2 = 0$
$8y + x^2 - 8 = 0$

b) $y + 3x - 6 = 0$
$y = x^2 + 4 - 4x$

18. A stone is thrown vertically upwards and its height (in metres) above the ground at time t (in seconds) is given by:

$h(t) = 35 - 5t^2 + 30t$

Find its initial height above the ground.

19. After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:

$$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

Assume that the petrol costs R 4,00 per litre and the driver earns R 18,00 per hour of travel time. Now deduce that the total cost, C, in Rands, for a 2000 km trip is given by:

$$C(x) = \frac{256\ 000}{x} + 40x$$

20. Solve the following quadratic equations by either factorisation, completing the square or by using the quadratic formula:

- Always try to factorise first, then use the formula if the trinomial cannot be factorised.
- Solve some of the equations by completing the square.

a) $-4y^2 - 41y - 45 = 0$

b) $16x^2 + 20x = 36$

c) $42p^2 + 104p + 64 = 0$

d) $21y + 3 = 54y^2$

e) $36y^2 + 44y + 8 = 0$

f) $12y^2 - 14 = 22y$

g) $16y^2 + 0y - 81 = 0$

h) $3y^2 + 10y - 48 = 0$

i) $63 - 5y^2 = 26y$

j) $2x^2 - 30 = 2$

k) $2y^2 = 98$

21. One root of the equation $9y^2 + 32 = ky$ is 8. Determine the value of k and the other root.

22. a) Solve for x in $x^2 - x = 6$.

    b) Hence, solve for y in $(y^2 - y)^2 - (y^2 - y) - 6 = 0$.

23. Solve for x: $x = \sqrt{8 - x} + 2$

24. a) Solve for y in $-4y^2 + 8y - 3 = 0$.

    b) Hence, solve for p in $4(p - 3)^2 - 8(p - 3) + 3 = 0$.

25. Solve for x: $2(x + 3)^{\frac{1}{2}} = 9$

26. a) Without solving the equation $x + \frac{1}{x} = 3$, determine the value of $x^2 + \frac{1}{x^2}$.

    b) Now solve $x + \frac{1}{x} = 3$ and use the result to assess the answer obtained in the question above.

27. Solve for y: $5(y - 1)^2 - 5 = 19 - (y - 1)^2$

28. Solve for t: $2t(t - \frac{3}{2}) = \frac{3}{2t^2 - 3t} + 2$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22CG	2. 22CH	3. 22CJ	4. 22CK	5a. 22CM	5b. 22CN
5c. 22CP	5d. 22CQ	6. 22CR	7. 22CS	8. 22CT	9. 22CV
10. 22CW	11. 22CX	12. 22CY	13. 22CZ	14. 22D2	15a. 22D3
15b. 22D4	15c. 22D5	15d. 22D6	15e. 22D7	15f. 22D8	15g. 22D9
15h. 22DB	15i. 22DC	15j. 22DD	16a. 22DF	16b. 22DG	16c. 22DH
16d. 22DJ	16e. 22DK	16f. 22DM	16g. 22DN	16h. 22DP	17a. 22DQ
17b. 22DR	18. 22DS	19. 22DT	20a. 22DV	20b. 22DW	20c. 22DX
20d. 22DY	20e. 22DZ	20f. 22F2	20g. 22F3	20h. 22F4	20i. 22F5
20j. 22F6	20k. 22F7	21. 22F8	22. 22F9	23. 22FB	24. 22FC
25. 22FD	26. 22FF	27. 22FG	28. 22FH		

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CHAPTER 3

Number patterns

In earlier grades we learned about linear sequences, where the difference between consecutive terms is constant. In this chapter, we will learn about quadratic sequences, where the difference between consecutive terms is not constant, but follows its own pattern.

3.1 Revision EMBG2

Terminology:	
Sequence/pattern	A sequence or pattern is an ordered set of numbers or variables.
Successive/consecutive	Successive or consecutive terms are terms that directly follow one after another in a sequence.
Common difference	The common or constant difference (d) is the difference between any two consecutive terms in a linear sequence.
General term	A mathematical expression that describes the sequence and that generates any term in the pattern by substituting different values for n.
Conjecture	A statement, consistent with known data, that has not been proved true nor shown to be false.

Important: a series is not the same as a sequence or pattern. Different types of series are studied in Grade 12. In Grade 11 we study sequences only.

See video: 22FJ at www.everythingmaths.co.za

Describing patterns EMBG3

To describe terms in a pattern we use the following notation:

- T_1 is the first term of a sequence.
- T_4 is the fourth term of a sequence.
- T_n is the general term and is often expressed as the n^{th} term of a sequence.

A sequence does not have to follow a pattern but when it does, we can write an equation for the general term. The general term can be used to calculate any term in the sequence. For example, consider the following linear sequence: $1; 4; 7; 10; 13; \ldots$ The n^{th} term is given by the equation $T_n = 3n - 2$.

You can check this by substituting values for n:

$$\begin{aligned}
T_1 &= 3(1) - 2 = 1 \\
T_2 &= 3(2) - 2 = 4 \\
T_3 &= 3(3) - 2 = 7 \\
T_4 &= 3(4) - 2 = 10 \\
T_5 &= 3(5) - 2 = 13
\end{aligned}$$

If we find the relationship between the position of a term and its value, we can describe the pattern and find any term in the sequence.

See video: 22FK at www.everythingmaths.co.za

Linear sequences EMBG4

DEFINITION: *Linear sequence*

A sequence of numbers in which there is a common difference (d) between any term and the term before it is called a linear sequence.

Important: $d = T_2 - T_1$, not $T_1 - T_2$.

Worked example 1: Linear sequence

QUESTION

Determine the common difference (d) and the general term for the following sequence:

$$10; 7; 4; 1; \ldots$$

SOLUTION

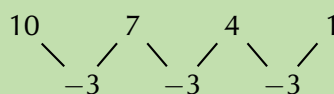
Step 1: Determine the common difference

To calculate the common difference, we find the difference between any term and the previous term:

$$d = T_n - T_{n-1}$$

$$\begin{aligned} \text{Therefore } d &= T_2 - T_1 \\ &= 7 - 10 \\ &= -3 \\ \text{or } d &= T_3 - T_2 \\ &= 4 - 7 \\ &= -3 \\ \text{or } d &= T_4 - T_3 \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

10 7 4 1

−3 −3 −3

Step 2: Determine the general term

To find the general term T_n, we must identify the relationship between:

- the **value** of a number in the pattern and
- the **position** of a number in the pattern

position	1	2	3	4
value	10	7	4	1

We start with the value of the first term in the sequence. We need to write an expression that includes the value of the common difference ($d = -3$) and the position of the term ($n = 1$).

$$\begin{aligned} T_1 &= 10 \\ &= 10 + (0)(-3) \\ &= 10 + (1-1)(-3) \end{aligned}$$

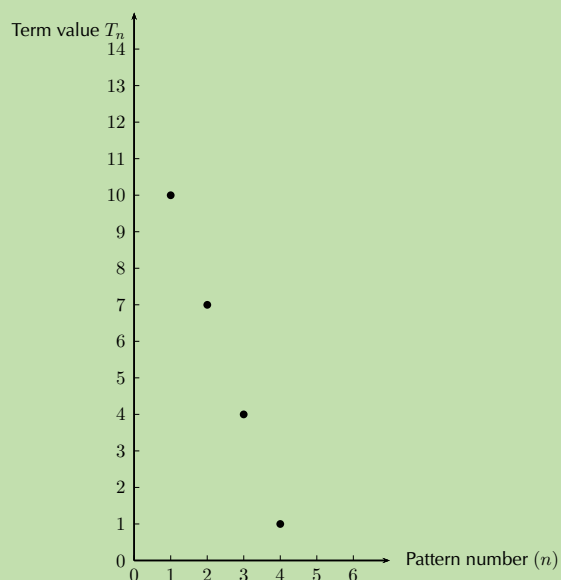
Now we write a similar expression for the second term.

$$\begin{aligned} T_2 &= 7 \\ &= 10 + (1)(-3) \\ &= 10 + (2-1)(-3) \end{aligned}$$

We notice a pattern forming that links the position of a number in the sequence to its value.

$$\begin{aligned} T_n &= 10 + (n-1)(-3) \\ &= 10 - 3n + 3 \\ &= -3n + 13 \end{aligned}$$

Step 3: Drawing a graph of the pattern

We can also represent this pattern graphically, as shown below.

Notice that the position numbers (n) can be positive integers only.

This pattern can also be expressed in words: "each term in the sequence can be calculated by multiplying negative three and the position number, and then adding thirteen."

See video: 22FM at www.everythingmaths.co.za

Exercise 3 – 1: Linear sequences

1. Write down the next three terms in each of the following sequences: $45; 29; 13; -3; \ldots$

2. The general term is given for each sequence below. Calculate the missing terms.

   a) $-4; -9; -14; \ldots; -24$
      $T_n = 1 - 5n$

   b) $6; \ldots; 24; \ldots; 42$
      $T_n = 9n - 3$

3. Find the general formula for the following sequences and then find T_{10}, T_{15} and T_{30}:

   a) $13; 16; 19; 22; \ldots$

   b) $18; 24; 30; 36; \ldots$

   c) $-10; -15; -20; -25; \ldots$

4. The seating in a classroom is arranged so that the first row has 20 desks, the second row has 22 desks, the third row has 24 desks and so on. Calculate how many desks are in the ninth row.

5. a) Complete the following:

$$13 + 31 = \ldots$$
$$24 + 42 = \ldots$$
$$38 + 83 = \ldots$$

   b) Look at the numbers on the left-hand side, what do you notice about the unit digit and the tens-digit?

   c) Investigate the pattern by trying other examples of 2-digit numbers.

   d) Make a conjecture about the pattern that you notice.

   e) Prove this conjecture.

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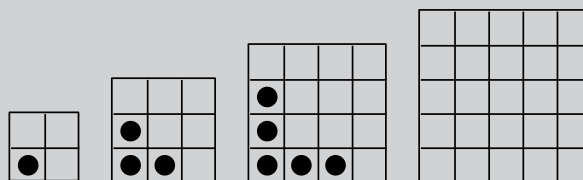
1. 22FN 2a. 22FP 2b. 22FQ 3a. 22FR 3b. 22FS 3c. 22FT
4. 22FV 5. 22FW

3.2 Quadratic sequences EMBG5

Investigation: Quadratic sequences

1. Study the dotted-tile pattern shown and answer the following questions.

   a) Complete the fourth pattern in the diagram.

   b) Complete the table below:

pattern number	1	2	3	4	5	20	n
dotted tiles	1	3	5				
difference (d)	–	2					

   c) What do you notice about the change in number of dotted tiles?

   d) Describe the pattern in words: "The number of dotted tiles...".

   e) Write the general term: $T_n = \ldots$

   f) Give the mathematical name for this kind of pattern.

   g) A pattern has 819 dotted tiles. Determine the value of n.

2. Now study the number of blank tiles (tiles without dots) and answer the following questions:

   a) Complete the table below:

pattern number	1	2	3	4	5	10
blank tiles	3	6	11			
first difference	–	3				
second difference	–	–				

   b) What do you notice about the change in the number of blank tiles?

   c) Describe the pattern in words: "The number of blank tiles...".

   d) Write the general term: $T_n = \ldots$

   e) Give the mathematical name for this kind of pattern.

   f) A pattern has 227 blank tiles. Determine the value of n.

   g) A pattern has 79 dotted tiles. Determine the number of blank tiles.

DEFINITION: *Quadratic sequence*

A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant.

Consider the following example: $1; 2; 4; 7; 11; \ldots$

The first difference is calculated by finding the difference between consecutive terms:

The second difference is obtained by taking the difference between consecutive first differences:

1 2 3 4
+1 +1 +1

We notice that the second differences are all equal to 1. Any sequence that has a common second difference is a **quadratic sequence**.

It is important to note that the first differences of a quadratic sequence form a sequence. This sequence has a constant difference between consecutive terms. In other words, a linear sequence results from taking the first differences of a quadratic sequence.

General case

If the sequence is quadratic, the n^{th} term is of the form $T_n = an^2 + bn + c$.

In each case, the common second difference is a $2a$.

Exercise 3 – 2: Quadratic sequences

1. Determine the second difference between the terms for the following sequences:

   a) $5; 20; 45; 80; \ldots$
   b) $6; 11; 18; 27; \ldots$
   c) $1; 4; 9; 16; \ldots$
   d) $3; 0; -5; -12; \ldots$
   e) $1; 3; 7; 13; \ldots$
   f) $0; -6; -16; -30; \ldots$
   g) $-1; 2; 9; 20; \ldots$
   h) $1; -3; -9; -17; \ldots$
   i) $3a+1; 12a+1; 27a+1; 48a+1 \ldots$
   j) $2; 10; 24; 44; \ldots$
   k) $t-2; 4t-1; 9t; 16t+1; \ldots$

2. Complete the sequence by filling in the missing term:

a) $11; 21; 35; \ldots; 75$

b) $20; \ldots; 42; 56; 72$

c) $\ldots; 37; 65; 101$

d) $3; \ldots; -13; -27; -45$

e) $24; 35; 48; \ldots; 80$

f) $\ldots; 11; 26; 47$

3. Use the general term to generate the first four terms in each sequence:

a) $T_n = n^2 + 3n - 1$

b) $T_n = -n^2 - 5$

c) $T_n = 3n^2 - 2n$

d) $T_n = -2n^2 + n + 1$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22FX 1b. 22FY 1c. 22FZ 1d. 22G2 1e. 22G3 1f. 22G4
1g. 22G5 1h. 22G6 1i. 22G7 1j. 22G8 1k. 22G9 2a. 22GB
2b. 22GC 2c. 22GD 2d. 22GF 2e. 22GG 2f. 22GH 3a. 22GJ
3b. 22GK 3c. 22GM 3d. 22GN

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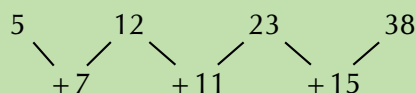
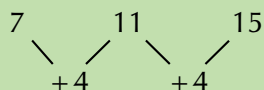
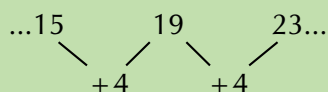
Worked example 2: Quadratic sequences

QUESTION

Write down the next two terms and determine an equation for the n^{th} term of the sequence 5; 12; 23; 38; ...

SOLUTION

Step 1: Find the first differences between the terms

Step 2: Find the second differences between the terms

So there is a common second difference of 4. We can therefore conclude that this is a quadratic sequence of the form $T_n = an^2 + bn + c$.

Continuing the sequence, the next first differences will be:

Step 3: Finding the next two terms in the sequence

The next two terms will be:

...38 57 80...

+19 +23

So the sequence will be: 5; 12; 23; 38; 57; 80; ...

Step 4: Determine the general term for the sequence

To find the values of a, b and c for $T_n = an^2 + bn + c$ we look at the first 3 terms in the sequence:

$$\begin{aligned} n = 1 &: T_1 = a + b + c \\ n = 2 &: T_2 = 4a + 2b + c \\ n = 3 &: T_3 = 9a + 3b + c \end{aligned}$$

We solve a set of simultaneous equations to determine the values of a, b and c

We know that $T_1 = 5$, $T_2 = 12$ and $T_3 = 23$

$$\begin{aligned} a + b + c &= 5 \\ 4a + 2b + c &= 12 \\ 9a + 3b + c &= 23 \end{aligned}$$

$$\begin{aligned} T_2 - T_1 &= 4a + 2b + c - (a + b + c) \\ 12 - 5 &= 4a + 2b + c - a - b - c \\ 7 &= 3a + b \qquad \ldots(1) \\ T_3 - T_2 &= 9a + 3b + c - (4a + 2b + c) \\ 23 - 12 &= 9a + 3b + c - 4a - 2b - c \\ 11 &= 5a + b \qquad \ldots(2) \end{aligned}$$

$$\begin{aligned} (2) - (1) &= 5a + b - (3a + b) \\ 11 - 7 &= 5a + b - 3a - b \\ 4 &= 2a \\ \therefore a &= 2 \end{aligned}$$

$$\begin{aligned} \text{Using equation } (1): \quad 3(2) + b &= 7 \\ \therefore b &= 1 \\ \text{And using} \quad a + b + c &= 5 \\ 2 + 1 + c &= 5 \\ \therefore c &= 1 \end{aligned}$$

Step 5: Write the general term for the sequence

$$T_n = 2n^2 + n + 2$$

Worked example 3: Plotting a graph of terms in a sequence

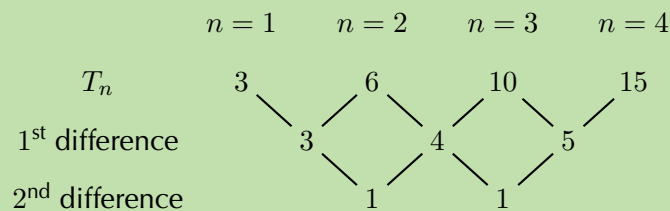
QUESTION

Consider the following sequence:

$$3; 6; 10; 15; 21; \ldots$$

1. Determine the general term (T_n) for the sequence.
2. Is this a linear or a quadratic sequence?
3. Plot a graph of T_n *vs* n.

SOLUTION

Step 1: Determine the first and second differences

We see that the first differences are not constant and form the sequence $3; 4; 5; \ldots$ and that there is a common second difference of 1. Therefore the sequence is quadratic and has a general term of the form $T_n = an^2 + bn + c$.

Step 2: Determine the general term T_n

To find the values of a, b and c for $T_n = an^2 + bn + c$ we look at the first 3 terms in the sequence:

$$\begin{aligned} n = 1 : T_1 &= a + b + c \\ n = 2 : T_2 &= 4a + 2b + c \\ n = 3 : T_3 &= 9a + 3b + c \end{aligned}$$

We solve this set of simultaneous equations to determine the values of a, b and c. We know that $T_1 = 3$, $T_2 = 6$ and $T_3 = 10$.

$$\begin{aligned} a + b + c &= 3 \\ 4a + 2b + c &= 6 \\ 9a + 3b + c &= 10 \end{aligned}$$

$$\begin{aligned} T_2 - T_1 &= 4a + 2b + c - (a + b + c) \\ 6 - 3 &= 4a + 2b + c - a - b - c \\ 3 &= 3a + b && \ldots (1) \end{aligned}$$

$$\begin{aligned} T_3 - T_2 &= 9a + 3b + c - (4a + 2b + c) \\ 10 - 6 &= 9a + 3b + c - 4a - 2b - c \\ 4 &= 5a + b && \ldots (2) \end{aligned}$$

$$\begin{aligned}(2)-(1) &= 5a+b-(3a+b)\\ 4-3 &= 5a+b-3a-b\\ 1 &= 2a\\ \therefore a &= \frac{1}{2}\end{aligned}$$

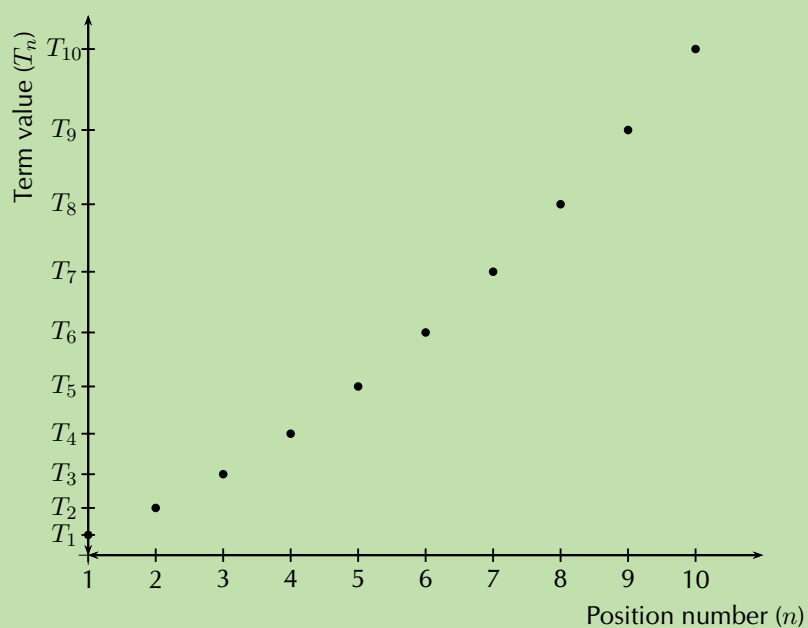
$$\begin{aligned}\text{Using equation (1)}: \quad 3\left(\frac{1}{2}\right)+b &= 3\\ \therefore b &= \frac{3}{2}\\ \text{And using} \quad a+b+c &= 3\\ \frac{1}{2}+\frac{3}{2}+c &= 3\\ \therefore c &= 1\end{aligned}$$

Therefore the general term for the sequence is $T_n = \frac{1}{2}n^2 + \frac{3}{2}n + 1$.

Step 3: Plot a graph of T_n vs n

Use the general term for the sequence, $T_n = \frac{1}{2}n^2 + \frac{3}{2}n + 1$, to complete the table.

n	1	2	3	4	5	6	7	8	9	10
T_n	3	6	10	15	21	28	36	45	55	66

Use the table to plot the graph:

In this case it would not be accurate to join these points, since n indicates the position of a term in a sequence and can therefore only be a positive integer. We can, however, see that the plot of the points lies in the shape of a parabola.

Worked example 4: Olympic Games soccer event

QUESTION

In the first stage of the soccer event at the Olympic Games, there are teams from four different countries in each group. Each country in a group must play every other country in the group once.

1. How many matches will be played in each group in the first stage of the event?
2. How many matches would be played if there are 5 teams in each group?
3. How many matches would be played if there are 6 teams in each group?
4. Determine the general formula of the sequence.

SOLUTION

Step 1: Determine the number of matches played if there are 4 teams in a group

Let the teams from four different countries be A, B, C and D.

teams in a group	matches played
A	AB, AC, AD
B	BC, BD
C	CD
D	
4	$3 + 2 + 1 = 6$

AB means that team A plays team B and BA would be the same match as AB. So if there are four different teams in a group, each group plays 6 matches.

Step 2: Determine the number of matches played if there are 5 teams in a group

Let the teams from five different countries be A, B, C, D and E.

teams in a group	matches played
A	AB, AC, AD, AE
B	BC, BD, BE
C	CD, CE
D	DE
E	
5	$4 + 3 + 2 + 1 = 10$

So if there are five different teams in a group, each group plays 10 matches.

Step 3: Determine the number of matches played if there are 6 teams in a group

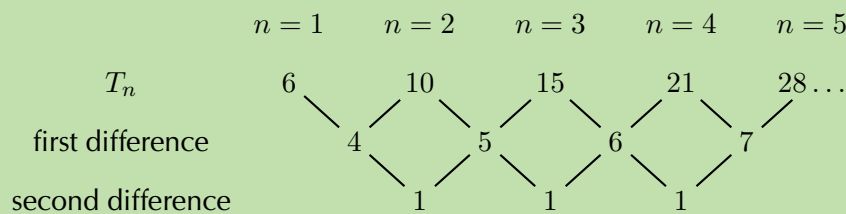
Let the teams from six different countries be A, B, C, D, E and F.

teams in a group	matches to be played
A	AB, AC, AD, AE, AF
B	BC, BD, BE, BF
C	CD, CE, CF
D	DE, DF
E	EF
F	
5	$5 + 4 + 3 + 2 + 1 = 15$

So if there are six different teams in a group, each group plays 15 matches.

We continue to increase the number of teams in a group and find that a group of 7 teams plays 21 matches and a group of 8 teams plays 28 matches.

Step 4: Consider the sequence

We examine the sequence to determine if it is linear or quadratic:

We see that the first differences are not constant and that there is a common second difference of 1. Therefore the sequence is quadratic and has a general term of the form $T_n = an^2 + bn + c$.

Step 5: Determine the general term T_n

To find the values of a, b and c for $T_n = an^2 + bn + c$ we look at the first 3 terms in the sequence:

$$\begin{aligned} n = 1 : T_1 &= a + b + c \\ n = 2 : T_2 &= 4a + 2b + c \\ n = 3 : T_3 &= 9a + 3b + c \end{aligned}$$

We solve a set of simultaneous equations to determine the values of a, b and c. We know that $T_1 = 6$, $T_2 = 10$ and $T_3 = 15$

$$\begin{aligned} a + b + c &= 6 \\ 4a + 2b + c &= 10 \\ 9a + 3b + c &= 15 \end{aligned}$$

$$\begin{aligned} T_2 - T_1 &= 4a + 2b + c - (a + b + c) \\ 10 - 6 &= 4a + 2b + c - a - b - c \\ 4 &= 3a + b \qquad \ldots(1) \end{aligned}$$

$$\begin{aligned} T_3 - T_2 &= 9a + 3b + c - (4a + 2b + c) \\ 15 - 10 &= 9a + 3b + c - 4a - 2b - c \\ 5 &= 5a + b \qquad \ldots(2) \end{aligned}$$

$$
\begin{aligned}
(2) - (1) &= 5a + b - (3a + b) \\
5 - 4 &= 5a + b - 3a - b \\
1 &= 2a \\
\therefore a &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Using equation } (1): \quad 3\left(\frac{1}{2}\right) + b &= 4 \\
\therefore b &= \frac{5}{2} \\
\text{And using } a + b + c &= 6 \\
\frac{1}{2} + \frac{5}{2} + c &= 6 \\
\therefore c &= 3
\end{aligned}
$$

Therefore the general term for the sequence is $T_n = \frac{1}{2}n^2 + \frac{5}{2}n + 3$.

Exercise 3 – 3: Quadratic sequences

1. Calculate the common second difference for each of the following quadratic sequences:

   a) $3;\ 6;\ 10;\ 15;\ 21;\ \ldots$

   b) $4;\ 9;\ 16;\ 25;\ 36;\ \ldots$

   c) $7;\ 17;\ 31;\ 49;\ 71;\ \ldots$

   d) $2;\ 10;\ 26;\ 50;\ 82;\ \ldots$

   e) $31;\ 30;\ 27;\ 22;\ 15;\ \ldots$

2. Find the first five terms of the quadratic sequence defined by: $T_n = 5n^2 + 3n + 4$.

3. Given $T_n = 4n^2 + 5n + 10$, find T_9.

4. Given $T_n = 2n^2$, for which value of n does $T_n = 32$?

5. a) Write down the next two terms of the quadratic sequence: $16; 27; 42; 61; \ldots$

   b) Find the general formula for the quadratic sequence above.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22GP 1b. 22GQ 1c. 22GR 1d. 22GS 1e. 22GT 2. 22GV
3. 22GW 4. 22GX 5. 22GY

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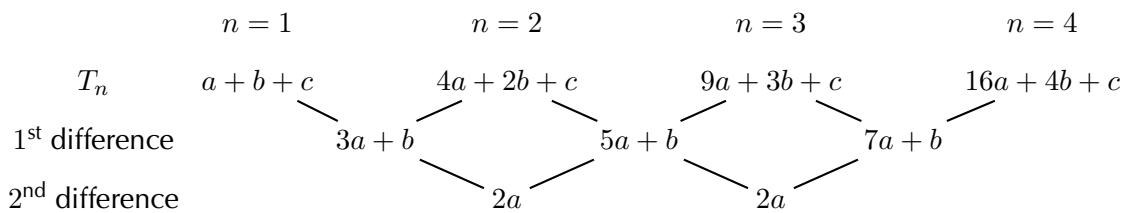
See presentation: 22GZ at www.everythingmaths.co.za

- T_n is the general term of a sequence.
- **Successive** or **consecutive** terms are terms that follow one after another in a sequence.
- A **linear** sequence has a common difference (d) between any two successive terms.

$$d = T_n - T_{n-1}$$

- A **quadratic** sequence has a common second difference between any two successive terms.
- The general term for a quadratic sequence is

$$T_n = an^2 + bn + c$$

- A general quadratic sequence:

Exercise 3 – 4: End of chapter exercises

1. Find the first five terms of the quadratic sequence defined by:
   $T_n = n^2 + 2n + 1$
2. Determine whether each of the following sequences is:
   - a linear sequence,
   - a quadratic sequence,
   - or neither.

   a) $6; 9; 14; 21; 30; \ldots$
   b) $1; 7; 17; 31; 49; \ldots$
   c) $8; 17; 32; 53; 80; \ldots$
   d) $9; 26; 51; 84; 125; \ldots$
   e) $2; 20; 50; 92; 146; \ldots$
   f) $5; 19; 41; 71; 109; \ldots$
   g) $2; 6; 10; 14; 18; \ldots$
   h) $3; 9; 15; 21; 27; \ldots$
   i) $1; 2{,}5; 5; 8{,}5; 13; \ldots$
   j) $10; 24; 44; 70; 102; \ldots$
   k) $2\frac{1}{2}; 6; 10\frac{1}{2}; 16; 22\frac{1}{2}; \ldots$
   l) $3p^2; 6p^2; 9p^2; 12p^2; 15p^2; \ldots$
   m) $2k; 8k; 18k; 32k; 50k; \ldots$

3. Given the pattern: $16; x; 46; \ldots$, determine the value of x if the pattern is linear.

4. Given $T_n = 2n^2$, for which value of n does $T_n = 242$?

5. Given $T_n = 3n^2$, find T_{11}.

6. Given $T_n = n^2 + 4$, for which value of n does $T_n = 85$?

7. Given $T_n = 4n^2 + 3n - 1$, find T_5.

8. Given $T_n = \frac{3}{2}n^2$, for which value of n does $T_n = 96$?

9. For each of the following patterns, determine:

   - the next term in the pattern,
   - and the general term,
   - the tenth term in the pattern.

   a) $3; 7; 11; 15; \ldots$
   b) $17; 12; 7; 2; \ldots$
   c) $\frac{1}{2}; 1; 1\frac{1}{2}; 2; \ldots$
   d) $a; a+b; a+2b; a+3b; \ldots$
   e) $1; -1; -3; -5; \ldots$

10. For each of the following sequences, find the equation for the general term and then use the equation to find T_{100}.

    a) $4; 7; 12; 19; 28; \ldots$

    b) $2; 8; 14; 20; 26; \ldots$

    c) $7; 13; 23; 37; 55; \ldots$

    d) $5; 14; 29; 50; 77; \ldots$

Given: $T_n = 3n - 1$

11. a) Write down the first five terms of the sequence.

    b) What do you notice about the difference between any two consecutive terms?

    c) Will this always be the case for a linear sequence?

Given the following sequence: $-15; -11; -7; \ldots; 173$

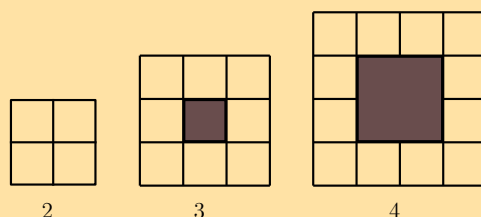
12. a) Determine the equation for the general term.

    b) Calculate how many terms there are in the sequence.

13. Given $3; 7; 13; 21; 31; \ldots$

    a) Thabang determines that the general term is $T_n = 4n - 1$. Is he correct? Explain.

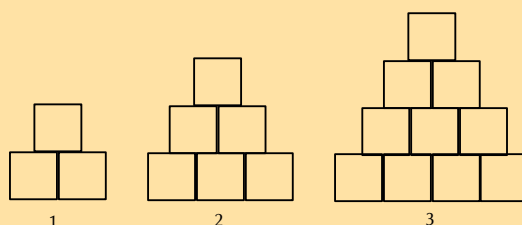
    b) Cristina determines that the general term is $T_n = n^2 + n + 1$. Is she correct? Explain.

14. Given the following pattern of blocks:

a) Draw pattern 5.

b) Complete the table below:

pattern number (n)	2	3	4	5	10	250	n
number of white blocks (w)	4	8					

c) Is this a linear or a quadratic sequence?

15. Cubes of volume 1 cm^3 are stacked on top of each other to form a tower:

a) Complete the table for the height of the tower:

tower number (n)	1	2	3	4	10	n
height of tower (h)	2					

b) What type of sequence is this?

c) Now consider the number of cubes in each tower and complete the table below:

tower number (n)	1	2	3	4
number of cubes (c)	3			

d) What type of sequence is this?

e) Determine the general term for this sequence.

f) How many cubes are needed for tower number 21?

g) How high will a tower of 496 cubes be?

16. A quadratic sequence has a second term equal to 1, a third term equal to -6 and a fourth term equal to -14.

a) Determine the second difference for this sequence.

b) Hence, or otherwise, calculate the first term of the pattern.

17. There are 15 schools competing in the U16 girls hockey championship and every team must play two matches — one home match and one away match.

    a) Use the given information to complete the table:

no. of schools	no. of matches
1	0
2	
3	
4	
5	

    b) Calculate the second difference.

    c) Determine a general term for the sequence.

    d) How many matches will be played if there are 15 schools competing in the championship?

    e) If 600 matches must be played, how many schools are competing in the championship?

18. The first term of a quadratic sequence is 4, the third term is 34 and the common second difference is 10. Determine the first six terms in the sequence.

19. Challenge question:

    Given that the general term for a quadratic sequences is $T_n = an^2 + bn + c$, let d be the first difference and D be the second common difference.

    a) Show that $a = \frac{D}{2}$.

    b) Show that $b = d - \frac{3}{2}D$.

    c) Show that $c = T_1 - d + D$.

    d) Hence, show that $T_n = \frac{D}{2}n^2 + \left(d - \frac{3}{2}D\right)n + (T_1 - d + D)$.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

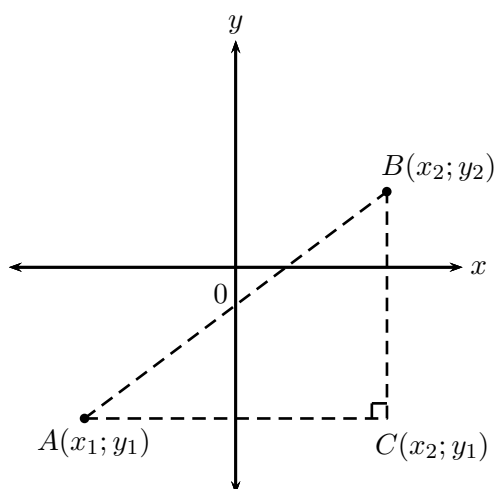
1. 22H2	2a. 22H3	2b. 22H4	2c. 22H5	2d. 22H6	2e. 22H7
2f. 22H8	2g. 22H9	2h. 22HB	2i. 22HC	2j. 22HD	2k. 22HF
2l. 22HG	2m. 22HH	3. 22HJ	4. 22HK	5. 22HM	6. 22HN
7. 22HP	8. 22HQ	9a. 22HR	9b. 22HS	9c. 22HT	9d. 22HV
9e. 22HW	10a. 22HX	10b. 22HY	10c. 22HZ	10d. 22J2	11. 22J3
12. 22J4	13. 22J5	14. 22J6	15. 22J7	16. 22J8	17. 22J9
18. 22JB	19. 22JC				

CHAPTER

Analytical geometry

Analytical geometry, also referred to as coordinate or Cartesian geometry, is the study of geometric properties and relationships between points, lines and angles in the Cartesian plane. Geometrical shapes are defined using a coordinate system and algebraic principles. In this chapter we deal with the equation of a straight line, parallel and perpendicular lines and inclination of a line.

4.1 Revision EMBG7

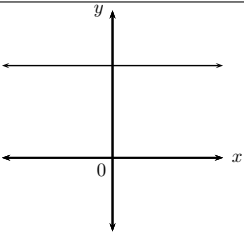
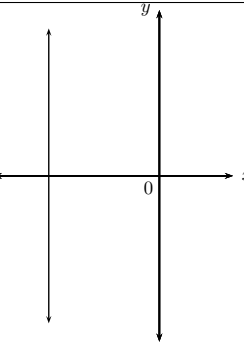
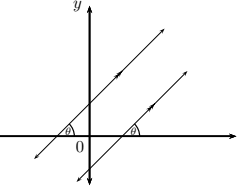
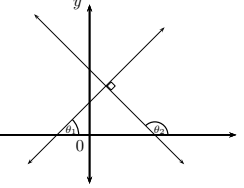
Points $A(x_1; y_1)$, $B(x_2; y_2)$ and $C(x_2; y_1)$ are shown in the diagram below:

Theorem of Pythagoras

$$AB^2 = AC^2 + BC^2$$

Distance formula

Distance between two points:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that $(x_1 - x_2)^2 = (x_2 - x_1)^2$.

See video: 22JD at www.everythingmaths.co.za

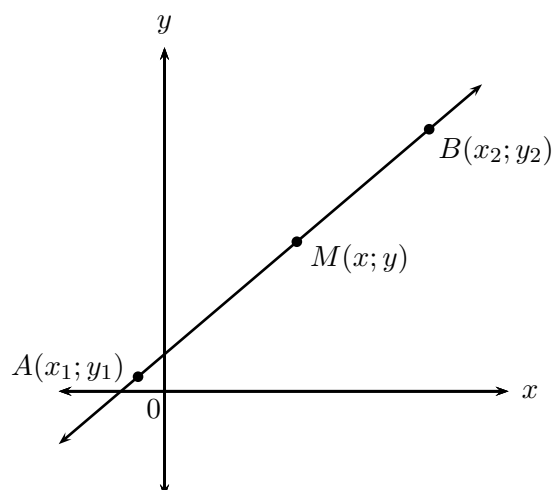
Gradient

Gradient (m) describes the slope or steepness of the line joining two points. The gradient of a line is determined by the ratio of vertical change to horizontal change.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$$

Remember to be consistent: $m \neq \frac{y_1 - y_2}{x_2 - x_1}$.

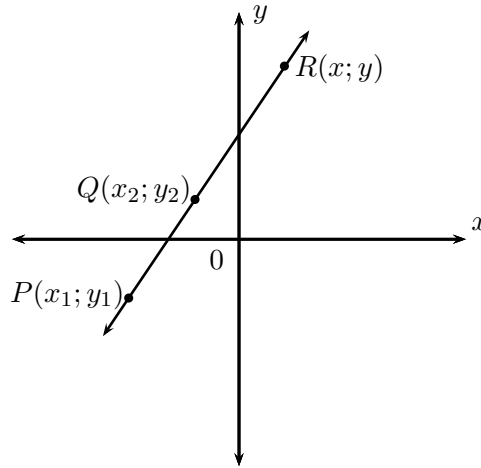
Horizontal lines		$m = 0$
Vertical lines		m is undefined
Parallel lines		$m_1 = m_2$
Perpendicular lines		$m_1 \times m_2 = -1$

Mid-point of a line segment

The coordinates of the mid-point $M(x; y)$ of a line between any two points $A(x_1; y_1)$ and $B(x_2; y_2)$:

$$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

See video: 22JF at www.everythingmaths.co.za

Points on a straight line

The diagram shows points $P(x_1; y_1)$, $Q(x_2; y_2)$ and $R(x; y)$ on a straight line.

We know that $m_{PR} = m_{QR} = m_{PQ}$.

Using $m_{PR} = m_{PQ}$, we obtain the following for any point $(x; y)$ on a straight line

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

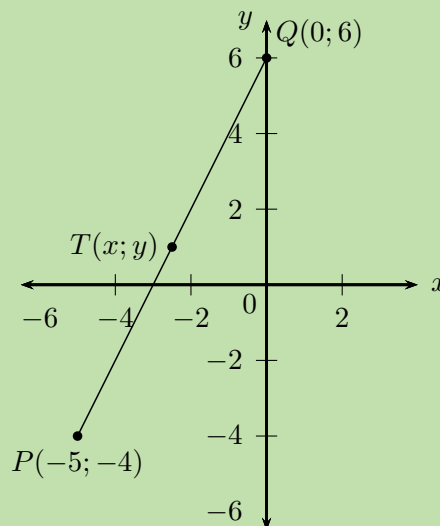
Worked example 1: Revision

QUESTION

Given the points $P(-5; -4)$ and $Q(0; 6)$:

1. Determine the length of the line segment PQ.
2. Determine the mid-point $T(x; y)$ of the line segment PQ.
3. Show that the line passing through $R(1; -\frac{3}{4})$ and $T(x; y)$ is perpendicular to the line PQ.

SOLUTION

Step 1: Draw a sketch

Step 2: Assign variables to the coordinates of the given points

Let the coordinates of P be $(x_1; y_1)$ and $Q\ (x_2; y_2)$

$$x_1 = -5; \qquad y_1 = -4; \qquad x_2 = 0; \qquad y_2 = 6$$

Write down the distance formula

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(0 - (-5))^2 + (6 - (-4))^2} \\ &= \sqrt{25 + 100} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

The length of the line segment PQ is $5\sqrt{5}$ units.

Step 3: Write down the mid-point formula and substitute the values

$$\begin{aligned} T(x; y) &= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) \\ x &= \frac{x_1 + x_2}{2} \\ &= \frac{-5 + 0}{2} \\ &= -\frac{5}{2} \\ y &= \frac{y_1 + y_2}{2} \\ &= \frac{-4 + 6}{2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

The mid-point of PQ is $T(-\frac{5}{2}; 1)$.

Step 4: Determine the gradients of PQ and RT

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} m_{PQ} &= \frac{6 - (-4)}{0 - (-5)} \\ &= \frac{10}{5} \\ &= 2 \end{aligned}$$

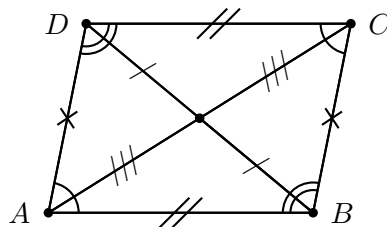
$$
\begin{aligned}
m_{RT} &= \frac{-\frac{3}{4} - 1}{1 - (-\frac{5}{2})} \\
&= \frac{-\frac{7}{4}}{\frac{7}{2}} \\
&= -\frac{7}{4} \times \frac{2}{7} \\
&= -\frac{1}{2}
\end{aligned}
$$

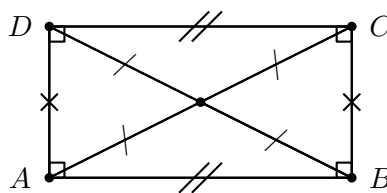
Calculate the product of the two gradients:

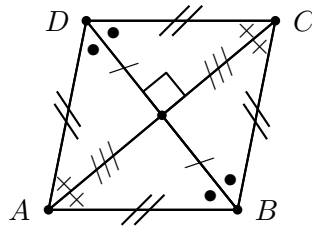
$$
\begin{aligned}
m_{RT} \times m_{PQ} &= -\frac{1}{2} \times 2 \\
&= -1
\end{aligned}
$$

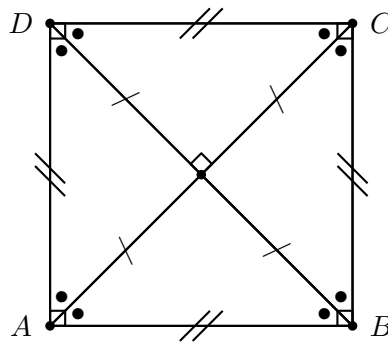
Therefore PQ is perpendicular to RT.

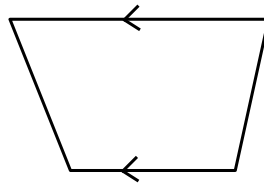
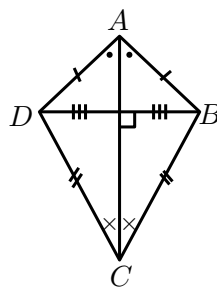
Quadrilaterals

- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other.

- A rectangle is a parallelogram that has all four angles equal to 90°.

  - Both pairs of opposite sides are equal and parallel.
  - The diagonals bisect each other.
  - The diagonals are equal in length.

- A rhombus is a parallelogram that has all four sides equal in length.

  - Both pairs of opposite sides are equal and parallel.
  - The diagonals bisect each other at 90°.
  - The diagonals of a rhombus bisect both pairs of opposite angles.

- A square is a rhombus that has all four interior angles equal to 90°.

  - Both pairs of opposite sides are equal and parallel.
  - The diagonals bisect each other at 90°.
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (that is, all angles are 45°).

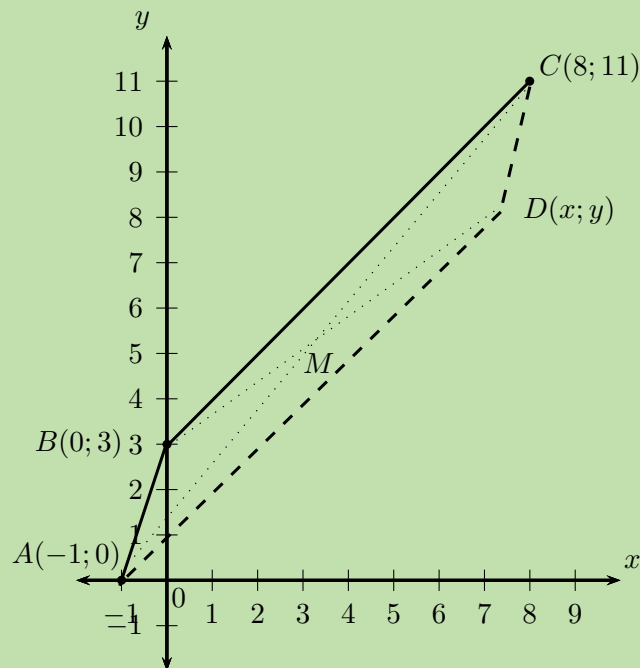
- A trapezium is a quadrilateral with one pair of opposite sides parallel.

- A kite is a quadrilateral with two pairs of adjacent sides equal.

  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.
  - The diagonal between equal sides bisects the interior angles.
  - The diagonals intersect at 90°.

Worked example 2: Quadrilaterals

QUESTION

Points $A(-1;0)$, $B(0;3)$, $C(8;11)$ and $D(x;y)$ are points on the Cartesian plane. Determine $D(x;y)$ if $ABCD$ is a parallelogram.

SOLUTION

Step 1: Draw a sketch

The mid-point of AC will be the same as the mid-point of BD. We first find the mid-point of AC and then use it to determine the coordinates of point D.

Step 2: Assign values to $(x_1;y_1)$ and $(x_2;y_2)$

Let the mid-point of AC be $M(x;y)$

$$x_1 = -1; \qquad y_1 = 0; \qquad x_2 = 8; \qquad y_2 = 11$$

Step 3: Write down the mid-point formula

$$M(x;y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

Step 4: Substitute the values and calculate the coordinates of M

$$\begin{aligned} M(x;y) &= \left(\frac{-1+8}{2}; \frac{0+11}{2}\right) \\ &= \left(\frac{7}{2}; \frac{11}{2}\right) \end{aligned}$$

Step 5: Use the coordinates of M to determine D

M is also the mid-point of BD so we use $M\left(\frac{7}{2};\frac{11}{2}\right)$ and $B\left(0;3\right)$ to find $D\left(x;y\right)$

Step 6: Substitute values and determine x and y

$$M = \left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$$
$$\therefore \left(\frac{7}{2};\frac{11}{2}\right) = \left(\frac{0+x}{2};\frac{3+y}{2}\right)$$

$$\frac{7}{2} = \frac{0+x}{2}$$
$$7 = 0+x$$
$$\therefore x = 7$$

$$\frac{11}{2} = \frac{3+y}{2}$$
$$11 = 3+y$$
$$\therefore y = 8$$

Step 7: Alternative method: inspection

Since we are given that $ABCD$ is a parallelogram, we can use the properties of a parallelogram and the given points to determine the coordinates of D.

From the sketch we expect that point D will lie below C.

Consider the given points A, B and C:

- Opposite sides of a parallelogram are parallel, therefore BC must be parallel to AD and their gradients must be equal.
- The vertical change from B to C is 8 units up.
- Therefore the vertical change from A to D is also 8 units up ($y = 0 + 8 = 8$).
- The horizontal change from B to C is 8 units to the right.
- Therefore the horizontal change from A to D is also 8 units to the right ($x = -1 + 8 = 7$).

or

- Opposite sides of a parallelogram are parallel, therefore AB must be parallel to DC and their gradients must be equal.
- The vertical change from A to B is 3 units up.

- Therefore the vertical change from C to D is 3 units down ($y = 11 - 3 = 8$).
- The horizontal change from A to B is 1 unit to the right.
- Therefore the horizontal change from C to D is 1 unit to the left ($x = 8 - 1 = 7$).

Step 8: Write the final answer

The coordinates of D are $(7; 8)$.

Exercise 4 – 1: Revision

1. Determine the length of the line segment between the following points:
   a) $P(-3; 5)$ and $Q(-1; -5)$
   b) $R(0{,}75; 3)$ and $S(0{,}75; -4)$
   c) $T(2x; y - 2)$ and $U(3x + 1; y - 2)$
2. Given $Q(4; 1), T(p; 3)$ and length $QT = \sqrt{8}$ units, determine the value of p.
3. Determine the gradient of the line AB if:
   a) $A(-5; 3)$ and $B(-7; 4)$
   b) $A(3; -2)$ and $B(1; -8)$
4. Prove that the line PQ, with $P(0; 3)$ and $Q(5; 5)$, is parallel to the line $5y + 5 = 2x$.
5. Given the points $A(-1; -1)$, $B(2; 5)$, $C(-1; -\frac{5}{2})$ and $D(x; -4)$ and $AB \perp CD$, determine the value of x.
6. Calculate the coordinates of the mid-point $P(x; y)$ of the line segment between the points:
   a) $M(3; 5)$ and $N(-1; -1)$
   b) $A(-3; -4)$ and $B(2; 3)$
7. The line joining $A(-2; 4)$ and $B(x; y)$ has the mid-point $C(1; 3)$. Determine the values of x and y.
8. Given quadrilateral $ABCD$ with vertices $A(0; 3), B(4; 3), C(5; -1)$ and $D(1; -1)$.
   a) Determine the equation of the line AD and the line BC.
   b) Show that $AD \parallel BC$.
   c) Calculate the lengths of AD and BC.
   d) Determine the equation of the diagonal BD.
   e) What type of quadrilateral is $ABCD$?

9. $MPQN$ is a parallelogram with points $M(-5;3)$, $P(-1;5)$ and $Q(4;5)$. Draw a sketch and determine the coordinates of $N(x;y)$.

10. $PQRS$ is a quadrilateral with points $P(-3;1)$, $Q(1;3)$, $R(6;1)$ and $S(2;-1)$ in the Cartesian plane.

    a) Determine the lengths of PQ and SR.
    b) Determine the mid-point of PR.
    c) Show that $PQ \parallel SR$.
    d) Determine the equations of the line PS and the line SR.
    e) Is $PS \perp SR$? Explain your answer.
    f) What type of quadrilateral is $PQRS$?

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22JG 1b. 22JH 1c. 22JJ 2. 22JK 3a. 22JM 3b. 22JN
4. 22JP 5. 22JQ 6a. 22JR 6b. 22JS 7. 22JT 8. 22JV
9. 22JW 10. 22JX

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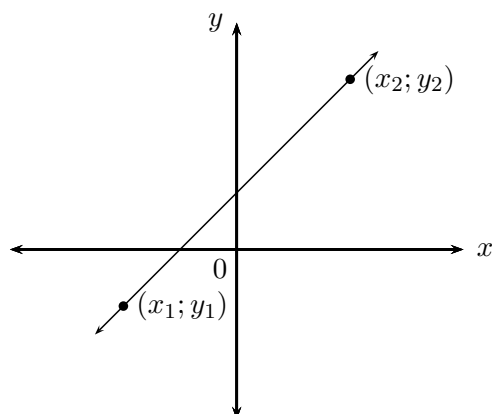
4.2 Equation of a line EMBG8

We can derive different forms of the straight line equation. The different forms are used depending on the information provided in the problem:

- The two-point form of the straight line equation: $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$
- The gradient–point form of the straight line equation: $y - y_1 = m(x - x_1)$
- The gradient–intercept form of the straight line equation: $y = mx + c$

The two-point form of the straight line equation EMBG9

Given any two points $(x_1; y_1)$ and $(x_2; y_2)$, we can determine the equation of the line passing through the two points using the equation:

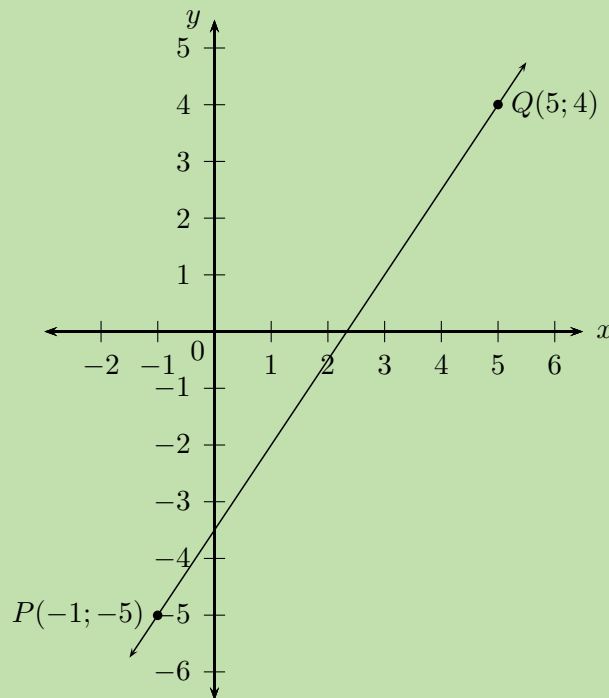
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Worked example 3: The two-point form of the straight line equation

QUESTION

Find the equation of the straight line passing through $P(-1; -5)$ and $Q(5; 4)$.

SOLUTION

Step 1: Draw a sketch

Step 2: Assign variables to the coordinates of the given points

Let the coordinates of P be $(x_1; y_1)$ and $Q(x_2; y_2)$

$$x_1 = -1; \qquad y_1 = -5; \qquad x_2 = 5; \qquad y_2 = 4$$

Step 3: Write down the two-point form of the straight line equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 4: Substitute the values and make y the subject of the equation

$$\begin{aligned}\frac{y-(-5)}{x-(-1)} &= \frac{4-(-5)}{5-(-1)}\\ \frac{y+5}{x+1} &= \frac{9}{6}\\ y+5 &= \frac{3}{2}(x+1)\\ y+5 &= \frac{3}{2}x+\frac{3}{2}\\ y &= \frac{3}{2}x-\frac{7}{2}\end{aligned}$$

Step 5: Write the final answer

$$y = \frac{3}{2}x - 3\frac{1}{2}$$

Exercise 4 – 2: The two-point form of the straight line equation

Determine the equation of the straight line passing through the points:

1. $(3;7)$ and $(-6;1)$
2. $(1;-\frac{11}{4})$ and $(\frac{2}{3};-\frac{7}{4})$
3. $(-2;1)$ and $(3;6)$
4. $(2;3)$ and $(3;5)$
5. $(1;-5)$ and $(-7;-5)$
6. $(-4;0)$ and $(1;\frac{15}{4})$
7. $(s;t)$ and $(t;s)$
8. $(-2;-8)$ and $(1;7)$
9. $(2p;q)$ and $(0;-q)$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22JY 2. 22JZ 3. 22K2 4. 22K3 5. 22K4 6. 22K5
7. 22K6 8. 22K7 9. 22K8

The gradient–point form of the straight line equation EMBGB

We derive the gradient–point form of the straight line equation using the definition of gradient and the two-point form of a straight line equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute $m = \frac{y_2 - y_1}{x_2 - x_1}$ on the right-hand side of the equation

$$\frac{y - y_1}{x - x_1} = m$$

Multiply both sides of the equation by $(x - x_1)$

$$y - y_1 = m(x - x_1)$$

To use this equation, we need to know the gradient of the line and the coordinates of one point on the line.

See video: 22K9 at www.everythingmaths.co.za

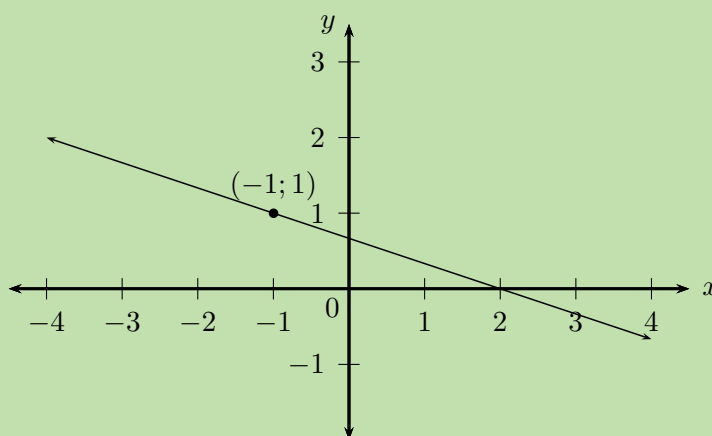
Worked example 4: The gradient–point form of the straight line equation

QUESTION

Determine the equation of the straight line with gradient $m = -\frac{1}{3}$ and passing through the point $(-1; 1)$.

SOLUTION

Step 1: Draw a sketch

We notice that $m < 0$, therefore the graph decreases as x increases.

Step 2: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute the value of the gradient

$$y - y_1 = -\frac{1}{3}(x - x_1)$$

Substitute the coordinates of the given point

$$\begin{aligned} y - 1 &= -\frac{1}{3}(x - (-1)) \\ y - 1 &= -\frac{1}{3}(x + 1) \\ y &= -\frac{1}{3}x - \frac{1}{3} + 1 \\ &= -\frac{1}{3}x + \frac{2}{3} \end{aligned}$$

Step 3: Write the final answer

The equation of the straight line is $y = -\frac{1}{3}x + \frac{2}{3}$.

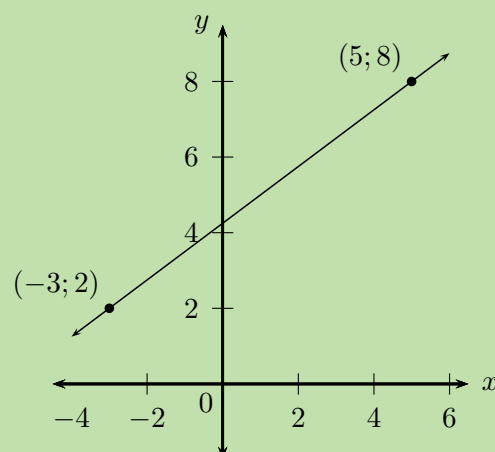
If we are given two points on a straight line, we can also use the gradient–point form to determine the equation of a straight line. We first calculate the gradient using the two given points and then substitute either of the two points into the gradient–point form of the equation.

Worked example 5: The gradient–point form of the straight line equation

QUESTION

Determine the equation of the straight line passing through $(-3; 2)$ and $(5; 8)$.

SOLUTION

Step 1: Draw a sketch

Step 2: Assign variables to the coordinates of the given points

$$x_1 = -3; \qquad y_1 = 2; \qquad x_2 = 5; \qquad y_2 = 8$$

Step 3: Calculate the gradient using the two given points

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 2}{5 - (-3)} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

Step 4: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute the value of the gradient

$$y - y_1 = \frac{3}{4}(x - x_1)$$

Substitute the coordinates of a given point

$$\begin{aligned} y - y_1 &= \frac{3}{4}(x - x_1) \\ y - 2 &= \frac{3}{4}(x - (-3)) \\ y - 2 &= \frac{3}{4}(x + 3) \\ y &= \frac{3}{4}x + \frac{9}{4} + 2 \\ &= \frac{3}{4}x + \frac{17}{4} \end{aligned}$$

Step 5: Write the final answer

The equation of the straight line is $y = \frac{3}{4}x + 4\frac{1}{4}$.

See video: 22KB at www.everythingmaths.co.za

Exercise 4 – 3: Gradient–point form of a straight line equation

Determine the equation of the straight line:

1. passing through the point $(-1; \frac{10}{3})$ and with $m = \frac{2}{3}$.
2. with $m = -1$ and passing through the point $(-2; 0)$.
3. passing through the point $(3; -1)$ and with $m = -\frac{1}{3}$.
4. parallel to the x-axis and passing through the point $(0; 11)$.
5. passing through the point $(1; 5)$ and with $m = -2$.
6. perpendicular to the x-axis and passing through the point $(-\frac{3}{2}; 0)$.
7. with $m = -0{,}8$ and passing through the point $(10; -7)$.
8. with undefined gradient and passing through the point $(4; 0)$.
9. with $m = 3a$ and passing through the point $(-2; -6a + b)$.

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1. 22KC 2. 22KD 3. 22KF 4. 22KG 5. 22KH 6. 22KJ
7. 22KK 8. 22KM 9. 22KN

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The gradient–intercept form of a straight line equation EMBGC

Using the gradient–point form, we can also derive the gradient–intercept form of the straight line equation.

Starting with the equation

$$y - y_1 = m(x - x_1)$$

Expand the brackets and make y the subject of the formula

$$\begin{aligned} y - y_1 &= mx - mx_1 \\ y &= mx - mx_1 + y_1 \\ y &= mx + (y_1 - mx_1) \end{aligned}$$

We define constant c such that $c = y_1 - mx_1$ so that we get the equation

$$y = mx + c$$

This is also called the **standard form** of the straight line equation.

Notice that when $x = 0$, we have

$$\begin{aligned} y &= m(0) + c \\ &= c \end{aligned}$$

Therefore c is the y-intercept of the straight line.

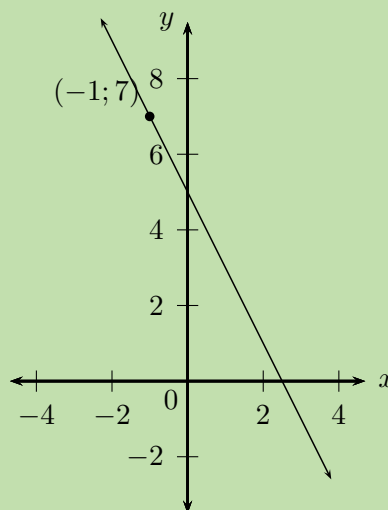
Worked example 6: The gradient–intercept form of straight line equation

QUESTION

Determine the equation of the straight line with gradient $m = -2$ and passing through the point $(-1; 7)$.

SOLUTION

Step 1: Slope of the line

We notice that $m < 0$, therefore the graph decreases as x increases.

Step 2: Write down the gradient–intercept form of straight line equation

$$y = mx + c$$

Substitute the value of the gradient

$$y = -2x + c$$

Substitute the coordinates of the given point and find c

$$\begin{aligned} y &= -2x + c \\ 7 &= -2(-1) + c \\ 7 - 2 &= c \\ \therefore c &= 5 \end{aligned}$$

This gives the y-intercept $(0; 5)$.

Step 3: Write the final answer

The equation of the straight line is $y = -2x + 5$.

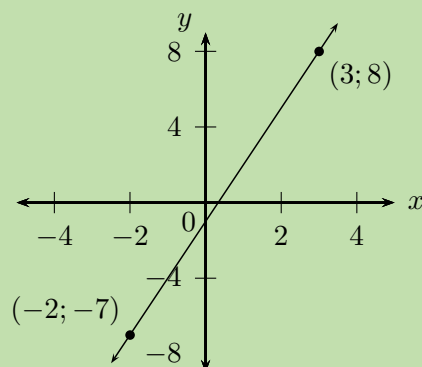
If we are given two points on a straight line, we can also use the gradient–intercept form to determine the equation of a straight line. We solve for the two unknowns m and c using simultaneous equations — using the methods of substitution or elimination.

Worked example 7: The gradient–intercept form of straight line equation

QUESTION

Determine the equation of the straight line passing through the points $(-2; -7)$ and $(3; 8)$.

SOLUTION

Step 1: Draw a sketch

Step 2: Write down the gradient–intercept form of straight line equation

$$y = mx + c$$

Step 3: Substitute the coordinates of the given points

$$
\begin{aligned}
-7 &= m(-2) + c \\
-7 &= -2m + c \qquad \ldots(1)
\end{aligned}
$$

$$
\begin{aligned}
8 &= m(3) + c \\
8 &= 3m + c \qquad \ldots(2)
\end{aligned}
$$

We have two equations with two unknowns; we can therefore solve using simultaneous equations.

Step 4: Make the coefficient of one of the variables the same in both equations

We notice that the coefficient of c in both equations is 1, therefore we can subtract one equation from the other to eliminate c:

$$
\begin{aligned}
-7 &= -2m + c \\
-(8 &= 3m + c) \\
-15 &= -5m \\
\therefore 3 &= m
\end{aligned}
$$

Substitute $m = 3$ into either of the two equations and determine c:

$$
\begin{aligned}
-7 &= -2m + c \\
-7 &= -2(3) + c \\
\therefore c &= -1
\end{aligned}
$$

or

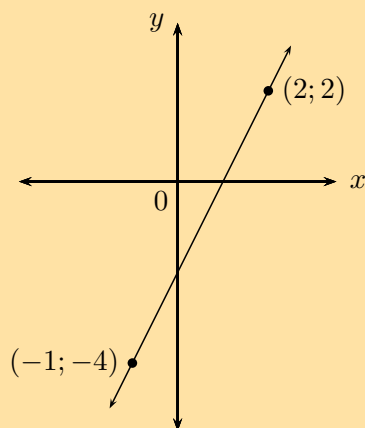
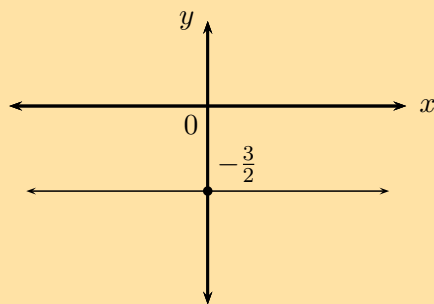
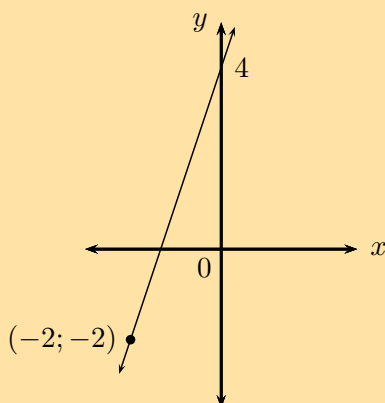
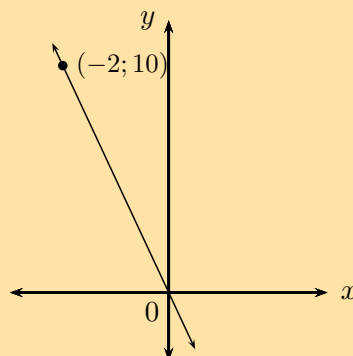
$$
\begin{aligned}
8 &= 3m + c \\
8 &= 3(3) + c \\
\therefore c &= -1
\end{aligned}
$$

Step 5: Write the final answer

The equation of the straight line is $y = 3x - 1$.

Exercise 4 – 4: The gradient–intercept form of a straight line equation

Determine the equation of the straight line:

1. passing through the point $(\frac{1}{2}; 4)$ and with $m = 2$.
2. passing through the points $(\frac{1}{2}; -2)$ and $(2; 4)$.
3. passing through the points $(2; -3)$ and $(-1; 0)$.
4. passing through the point $(2; -\frac{6}{7})$ and with $m = -\frac{3}{7}$.
5. which cuts the y-axis at $y = -\frac{1}{5}$ and with $m = \frac{1}{2}$.
6.

7.

8.

9.

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1. 22KP 2. 22KQ 3. 22KR 4. 22KS 5. 22KT 6. 22KV
7. 22KW 8. 22KX 9. 22KY

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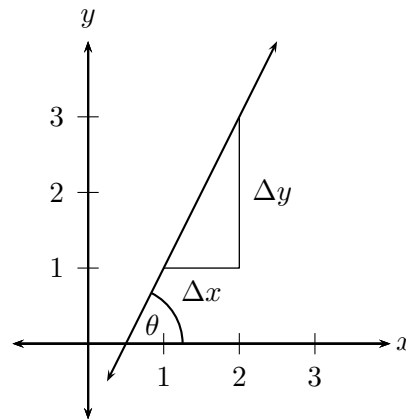
4.3 Inclination of a line EMBGD

The diagram shows that a straight line makes an angle θ with the positive x-axis. This is called the **angle of inclination** of a straight line.

We notice that if the gradient changes, then the value of θ also changes, therefore the angle of inclination of a line is related to its gradient. We know that gradient is the ratio of a change in the y-direction to a change in the x-direction:

$$m = \frac{\Delta y}{\Delta x}$$

From trigonometry we know that the tangent function is defined as the ratio:

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

And from the diagram we see that

$$\begin{aligned} \tan\theta &= \frac{\Delta y}{\Delta x} \\ \therefore m &= \tan\theta \qquad \text{for } 0^\circ \leq \theta < 180^\circ \end{aligned}$$

Therefore the gradient of a straight line is equal to the tangent of the angle formed between the line and the positive direction of the x-axis.

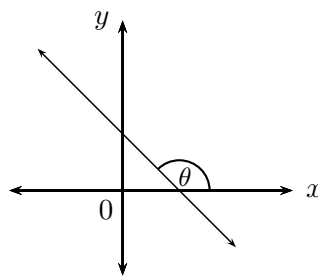
Vertical lines

- $\theta = 90^\circ$
- Gradient is undefined since there is no change in the x-values ($\Delta x = 0$).
- Therefore $\tan\theta$ is also undefined (the graph of $\tan\theta$ has an asymptote at $\theta = 90^\circ$).

Horizontal lines

- $\theta = 0°$
- Gradient is equal to 0 since there is no change in the y-values ($\Delta y = 0$).
- Therefore $\tan\theta$ is also equal to 0 (the graph of $\tan\theta$ passes through the origin $(0°; 0)$.

Lines with negative gradients

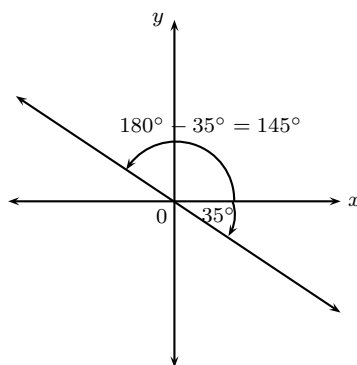
If a straight line has a negative gradient ($m < 0$, $\tan\theta < 0$), then the angle formed between the line and the positive direction of the x-axis is obtuse.

From the CAST diagram in trigonometry, we know that the tangent function is negative in the second and fourth quadrant. If we are calculating the angle of inclination for a line with a negative gradient, we must add 180° to change the negative angle in the fourth quadrant to an obtuse angle in the second quadrant:

If we are given a straight line with gradient $m = -0{,}7$, then we can determine the angle of inclination using a calculator:

$$\begin{aligned}
\tan\theta &= m \\
&= -0{,}7 \\
\therefore \theta &= \tan^{-1}(-0{,}7) \\
&= -35{,}0°
\end{aligned}$$

This negative angle lies in the fourth quadrant. We must add 180° to get an obtuse angle in the second quadrant:

$$\begin{aligned}
\theta &= -35{,}0° + 180° \\
&= 145°
\end{aligned}$$

And we can always use our calculator to check that the obtuse angle $\theta = 145°$ gives a gradient of $m = -0{,}7$.

Exercise 4 – 5: Angle of inclination

1. Determine the gradient (correct to 1 decimal place) of each of the following straight lines, given that the angle of inclination is equal to:

   a) 60°
   b) 135°
   c) 0°
   d) 54°
   e) 90°
   f) 45°
   g) 140°
   h) 180°
   i) 75°

2. Determine the angle of inclination (correct to 1 decimal place) for each of the following:

   a) a line with $m = \frac{3}{4}$
   b) $2y - x = 6$
   c) the line passes through the points $(-4; -1)$ and $(2; 5)$
   d) $y = 4$
   e) $x = 3y + \frac{1}{2}$
   f) $x = -0{,}25$
   g) the line passes through the points $(2; 5)$ and $(\frac{2}{3}; 1)$
   h) a line with gradient equal to 0,577

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1a. 22KZ 1b. 22M2 1c. 22M3 1d. 22M4 1e. 22M5 1f. 22M6
1g. 22M7 1h. 22M8 1i. 22M9 2a. 22MB 2b. 22MC 2c. 22MD
2d. 22MF 2e. 22MG 2f. 22MH 2g. 22MJ 2h. 22MK

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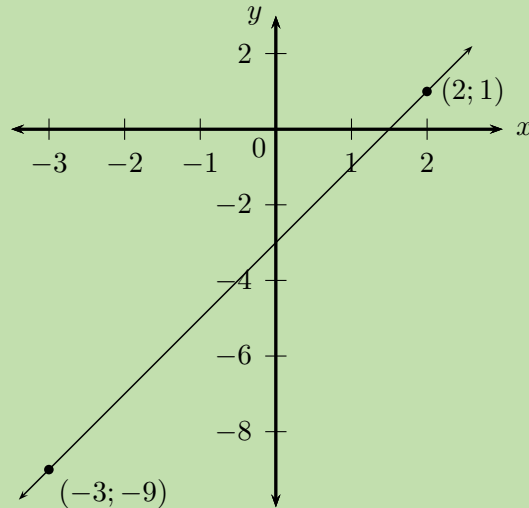
Worked example 8: Inclination of a straight line

QUESTION

Determine the angle of inclination (correct to 1 decimal place) of the straight line passing through the points $(2;1)$ and $(-3;-9)$.

SOLUTION

Step 1: Draw a sketch

Step 2: Assign variables to the coordinates of the given points

$$x_1 = 2; \qquad y_1 = 1; \qquad x_2 = -3; \qquad y_2 = -9$$

Step 3: Determine the gradient of the line

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-9 - 1}{-3 - 2} \\ &= \frac{-10}{-5} \\ \therefore m &= 2 \end{aligned}$$

Step 4: Use the gradient to determine the angle of inclination of the line

$$\begin{aligned} \tan\theta &= m \\ &= 2 \\ \therefore \theta &= \tan^{-1} 2 \\ &= 63{,}4^\circ \end{aligned}$$

Important: make sure your calculator is in DEG (degrees) mode.

Step 5: Write the final answer

The angle of inclination of the straight line is 63,4°.

Worked example 9: Inclination of a straight line

QUESTION

Determine the equation of the straight line passing through the point $(3;1)$ and with an angle of inclination of $135°$.

SOLUTION

Step 1: Use the angle of inclination to determine the gradient of the line

$$\begin{aligned} m &= \tan\theta \\ &= \tan 135° \\ \therefore m &= -1 \end{aligned}$$

Step 2: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute $m = -1$

$$y - y_1 = -(x - x_1)$$

Substitute the given point $(3;1)$

$$\begin{aligned} y - 1 &= -(x - 3) \\ y &= -x + 3 + 1 \\ &= -x + 4 \end{aligned}$$

Step 3: Write the final answer

The equation of the straight line is $y = -x + 4$.

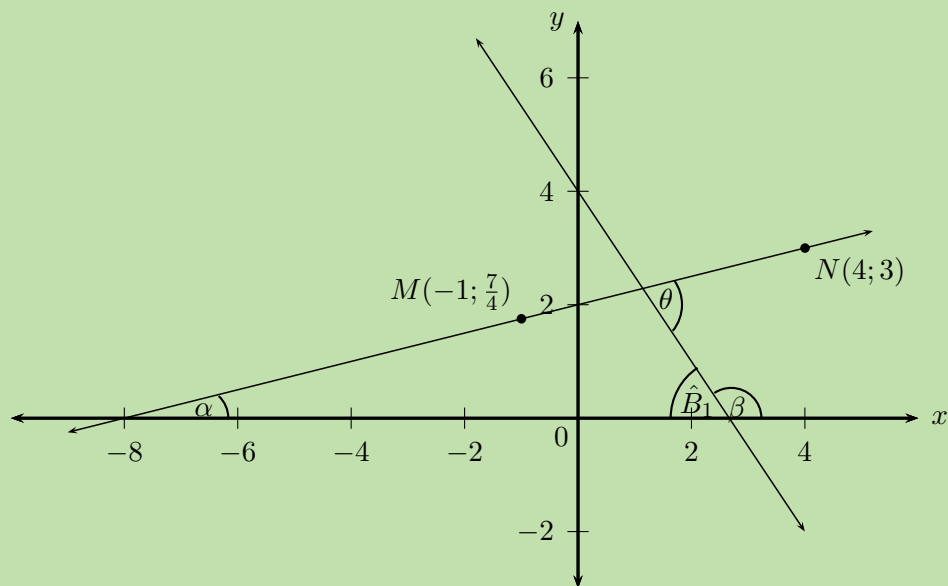
Worked example 10: Inclination of a straight line

QUESTION

Determine the acute angle (correct to 1 decimal place) between the line passing through the points $M(-1;1\frac{3}{4})$ and $N(4;3)$ and the straight line $y = -\frac{3}{2}x + 4$.

SOLUTION

Step 1: Draw a sketch

Draw the line through points $M(-1;1\frac{3}{4})$ and $N(4;3)$ and the line $y = -\frac{3}{2}x + 4$ on a suitable system of axes. Label α and β, the angles of inclination of the two lines. Label θ, the acute angle between the two straight lines.

Notice that α and θ are acute angles and β is an obtuse angle.

$$\begin{aligned}
\hat{B}_1 &= 180° - \beta && (\angle \text{ on str. line}) \\
\text{and } \theta &= \alpha + \hat{B}_1 && (\text{ext. } \angle \text{ of } \triangle = \text{ sum int. opp}) \\
\therefore \theta &= \alpha + (180° - \beta) \\
&= 180° + \alpha - \beta
\end{aligned}$$

Step 2: Use the gradient to determine the angle of inclination β

From the equation $y = -\frac{3}{2}x + 4$ we see that $m < 0$, therefore β is an obtuse angle such that $90° < \beta < 180°$.

$$\begin{aligned}
\tan\beta &= m \\
&= -\frac{3}{2} \\
\tan^{-1}\left(-\frac{3}{2}\right) &= -56{,}3°
\end{aligned}$$

This negative angle lies in the fourth quadrant. We know that the angle of inclination β is an obtuse angle that lies in the second quadrant, therefore

$$\begin{aligned}
\beta &= -56{,}3° + 180° \\
&= 123{,}7°
\end{aligned}$$

Step 3: Determine the gradient and angle of inclination of the line through M and N

Determine the gradient

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 - \frac{7}{4}}{4 - (-1)} \\
&= \frac{\frac{5}{4}}{5} \\
&= \frac{1}{4}
\end{aligned}
$$

Determine the angle of inclination

$$
\begin{aligned}
\tan\alpha &= m \\
&= \frac{1}{4} \\
\therefore \alpha &= \tan^{-1}\left(\frac{1}{4}\right) \\
&= 14{,}0^\circ
\end{aligned}
$$

Step 4: Write the final answer

$$
\begin{aligned}
\theta &= 180^\circ + \alpha - \beta \\
&= 180^\circ + 14{,}0^\circ - 123{,}7^\circ \\
&= 70{,}3^\circ
\end{aligned}
$$

The acute angle between the two straight lines is $70{,}3^\circ$.

Exercise 4 – 6: Inclination of a straight line

1. Determine the angle of inclination for each of the following:

   a) a line with $m = \frac{4}{5}$

   b) $x + y + 1 = 0$

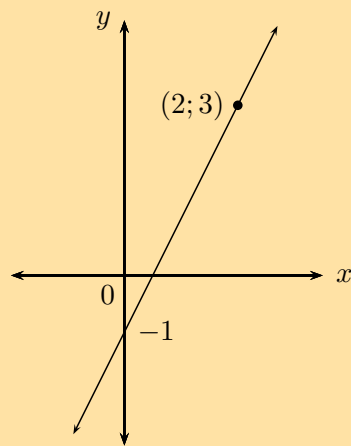
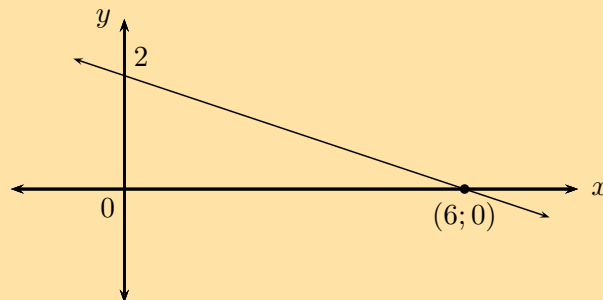
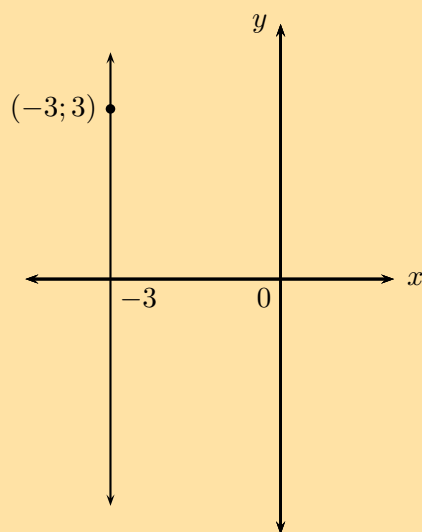
   c) a line with $m = 5{,}69$

   d) the line that passes through $(1;1)$ and $(-2;7)$

   e) $3 - 2y = 9x$

   f) the line that passes through $(-1;-6)$ and $(-\frac{1}{2};-\frac{11}{2})$

   g) $5 = 10y - 15x$

h)

i)

j)

2. Determine the acute angle between the line passing through the points $A(-2;\frac{1}{5})$ and $B(0;1)$ and the line passing through the points $C(1;0)$ and $D(-2;6)$.
3. Determine the angle between the line $y+x=3$ and the line $x=y+\frac{1}{2}$.
4. Find the angle between the line $y=2x$ and the line passing through the points $(-1;\frac{7}{3})$ and $(0;2)$.

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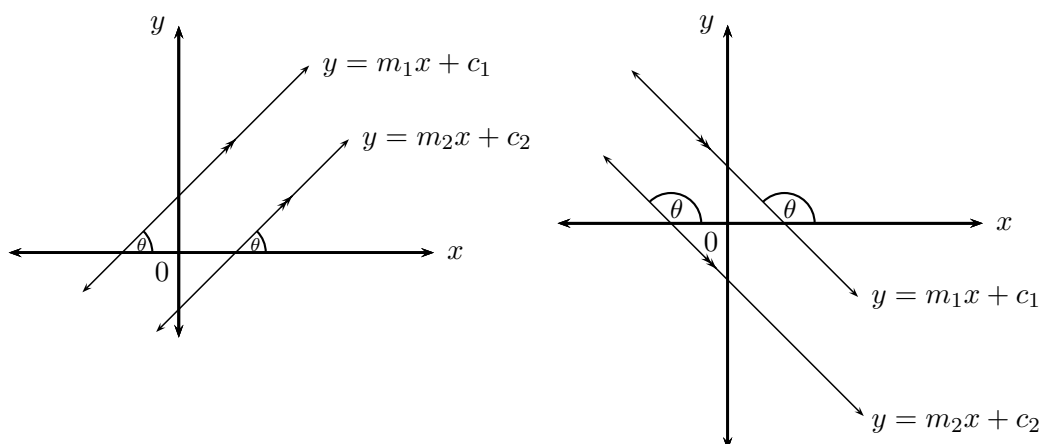
1a. 22MM 1b. 22MN 1c. 22MP 1d. 22MQ 1e. 22MR 1f. 22MS
1g. 22MT 1h. 22MV 1i. 22MW 1j. 22MX 2. 22MY 3. 22MZ
4. 22N2

4.4 Parallel lines EMBGF

Investigation: Parallel lines

1. Draw a sketch of the line passing through the points $P(-1;0)$ and $Q(1;4)$ and the line passing through the points $R(1;2)$ and $S(2;4)$.
2. Label and measure α and β, the angles of inclination of straight lines PQ and RS respectively.
3. Describe the relationship between α and β.
4. "α and β are alternate angles, therefore $PQ \parallel RS$." Is this a true statement? If not, provide a correct statement.
5. Use your calculator to determine $\tan\alpha$ and $\tan\beta$.
6. Complete the sentence: …… lines have …… angles of inclination.
7. Determine the equations of the straight lines PQ and RS.
8. What do you notice about m_{PQ} and m_{RS}?
9. Complete the sentence: …… lines have …… gradients.

Another method of determining the equation of a straight line is to be given a point on the unknown line, $(x_1;y_1)$, and the equation of a line which is parallel to the unknown line.

Let the equation of the unknown line be $y = m_1x + c_1$ and the equation of the given line be $y = m_2x + c_2$.

If the two lines are parallel then

$$m_1 = m_2$$

Important: when determining the gradient of a line using the coefficient of x, make sure the given equation is written in the gradient–intercept (standard) form. $y = mx + c$

Substitute the value of m_2 and the given point $(x_1; y_1)$, into the gradient–intercept form of a straight line equation

$$y - y_1 = m(x - x_1)$$

and determine the equation of the unknown line.

Worked example 11: Parallel lines

QUESTION

Determine the equation of the line that passes through the point $(-1; 1)$ and is parallel to the line $y - 2x + 1 = 0$.

SOLUTION

Step 1: Write the equation in gradient–intercept form

We write the given equation in gradient–intercept form and determine the value of m.

$$y = 2x - 1$$

We know that the two lines are parallel, therefore $m_1 = m_2 = 2$.

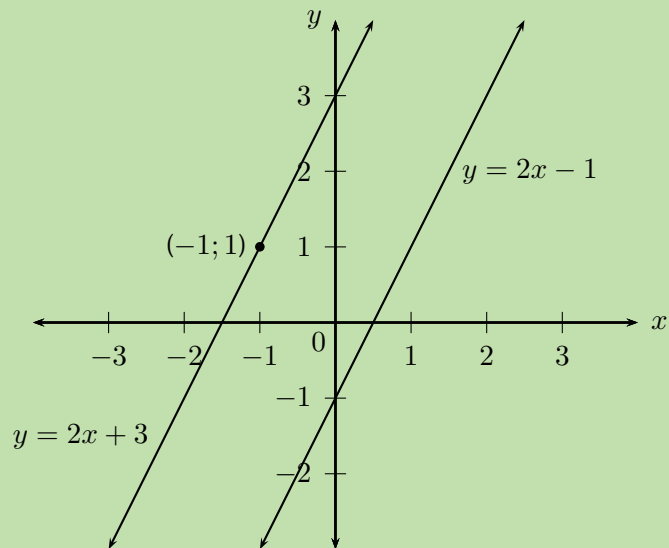
Step 2: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute $m = 2$

$$y - y_1 = 2(x - x_1)$$

Substitute the given point $(-1; 1)$

$$\begin{aligned} y - 1 &= 2(x - (-1)) \\ y - 1 &= 2x + 2 \\ y &= 2x + 2 + 1 \\ &= 2x + 3 \end{aligned}$$

A sketch was not required, but it is always helpful and can be used to check answers.

Step 3: Write the final answer

The equation of the straight line is $y = 2x + 3$.

Worked example 12: Parallel lines

QUESTION

Line AB passes through the point $A(0; 3)$ and has an angle of inclination of 153,4°. Determine the equation of the line CD which passes through the point $C(2; -3)$ and is parallel to AB.

SOLUTION

Step 1: Use the given angle of inclination to determine the gradient

$$\begin{aligned} m_{AB} &= \tan\theta \\ &= \tan 153{,}4^\circ \\ &= -0{,}5 \end{aligned}$$

Step 2: Parallel lines have equal gradients

Since we are given $AB \parallel CD$,

$$m_{CD} = m_{AB} = -0{,}5$$

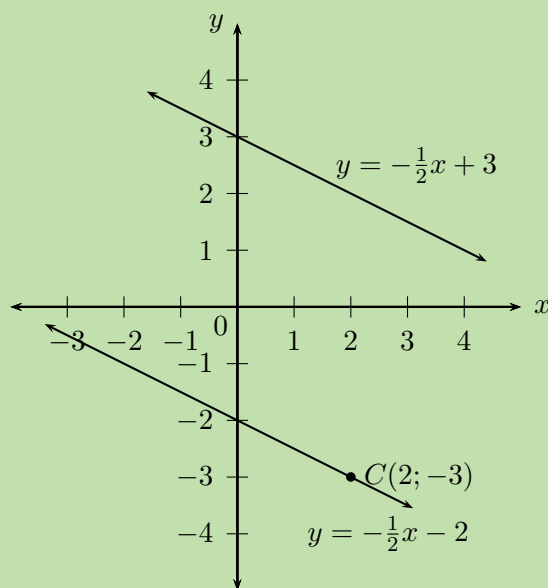
Step 3: Write down the gradient–point form of a straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute the gradient $m_{CD} = -0{,}5$.

$$y - y_1 = -\frac{1}{2}(x - x_1)$$

Substitute the given point $(2; -3)$.

$$\begin{aligned} y - (-3) &= -\frac{1}{2}(x - 2) \\ y + 3 &= -\frac{1}{2}x + 1 \\ y &= -\frac{1}{2}x - 2 \end{aligned}$$

A sketch was not required, but it is always useful.

Step 4: Write the final answer

The equation of the straight line is $y = -\frac{1}{2}x - 2$.

See video: 22N3 at www.everythingmaths.co.za

Exercise 4 – 7: Parallel lines

1. Determine whether or not the following two lines are parallel:

    a) $y + 2x = 1$ and $-2x + 3 = y$

    b) $\frac{y}{3} + x + 5 = 0$ and $2y + 6x = 1$

    c) $y = 2x - 7$ and the line passing through $(1; -2)$ and $(\frac{1}{2}; -1)$

    d) $y + 1 = x$ and $x + y = 3$

    e) The line passing through points $(-2; -1)$ and $(-4; -3)$ and the line $-y + x - 4 = 0$

    f) $y - 1 = \frac{1}{3}x$ and the line passing through points $(-2; 4)$ and $(1; 5)$

2. Determine the equation of the straight line that passes through the point $(1; -5)$ and is parallel to the line $y + 2x - 1 = 0$.

3. Determine the equation of the straight line that passes through the point $(-2; -6)$ and is parallel to the line $2y + 1 = 6x$.

4. Determine the equation of the straight line that passes through the point $(-2; -2)$ and is parallel to the line with angle of inclination $\theta = 56{,}31°$.

5. Determine the equation of the straight line that passes through the point $(-2; \frac{2}{5})$ and is parallel to the line with angle of inclination $\theta = 145°$.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22N4 1b. 22N5 1c. 22N6 1d. 22N7 1e. 22N8 1f. 22N9
2. 22NB 3. 22NC 4. 22ND 5. 22NF

4.5 Perpendicular lines

EMBGG

Investigation: Perpendicular lines

1. Draw a sketch of the line passing through the points $A(-2; -3)$ and $B(2; 5)$ and the line passing through the points $C(-1; \frac{1}{2})$ and $D(4; -2)$.

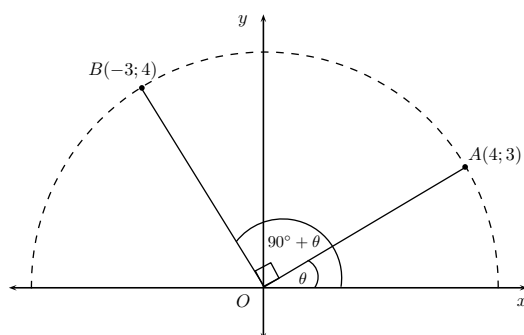
2. Label and measure α and β, the angles of inclination of straight lines AB and CD respectively.

3. Label and measure θ, the angle between the lines AB and CD.

4. Describe the relationship between the lines AB and CD.

5. "θ is a reflex angle, therefore $AB \perp CD$." Is this a true statement? If not, provide a correct statement.

6. Determine the equation of the straight line AB and the line CD.
7. Use your calculator to determine $\tan \alpha \times \tan \beta$.
8. Determine $m_{AB} \times m_{CD}$.
9. What do you notice about these products?
10. Complete the sentence: if two lines are ...... to each other, then the product of their ...... is equal ......
11. Complete the sentence: if the gradient of a straight line is equal to the negative ...... of the gradient of another straight line, then the two lines are ......

Deriving the formula: $m_1 \times m_2 = -1$

Consider the point $A(4;3)$ on the Cartesian plane with an angle of inclination $A\hat{O}X = \theta$. Rotate through an angle of 90° and place point B at $(-3;4)$ so that we have the angle of inclination $B\hat{O}X = 90° + \theta$.

We determine the gradient of OA:

$$\begin{aligned} m_{OA} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-0}{4-0} \\ &= \frac{3}{4} \end{aligned}$$

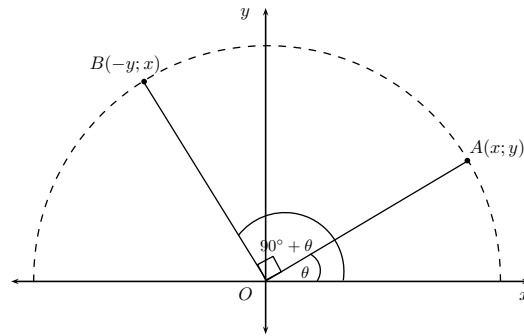
And determine the gradient of OB:

$$\begin{aligned} m_{OB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-0}{-3-0} \\ &= \frac{4}{-3} \end{aligned}$$

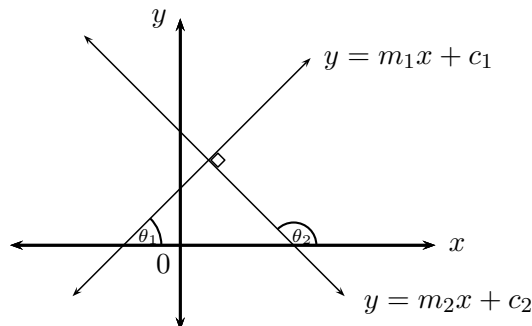
By rotating through an angle of 90° we know that $OB \perp OA$:

$$m_{OA} \times m_{OB} = \frac{3}{4} \times \frac{4}{-3}$$
$$= -1$$

We can also write that

$$m_{OA} = -\frac{1}{m_{OB}}$$

If we have the general point $A(x; y)$ with an angle of inclination $A\hat{O}X = \theta$ and point $B(-y; x)$ such that $B\hat{O}X = 90° + \theta$, then we know that

$$m_{OA} = \frac{y}{x}$$
$$m_{OB} = -\frac{x}{y}$$
$$\therefore m_{OA} \times m_{OB} = \frac{y}{x} \times -\frac{x}{y}$$
$$= -1$$

Another method of determining the equation of a straight line is to be given a point on the line, $(x_1; y_1)$, and the equation of a line which is perpendicular to the unknown line. Let the equation of the unknown line be $y = m_1 x + c_1$ and the equation of the given line be $y = m_2 x + c_2$.

If the two lines are perpendicular then

$$m_1 \times m_2 = -1$$

Note: this rule does not apply to vertical or horizontal lines.

When determining the gradient of a line using the coefficient of x, make sure the given equation is written in the gradient–intercept (standard) form $y = mx + c$. Then we know that

$$m_1 = -\frac{1}{m_2}$$

Substitute the value of m_1 and the given point $(x_1; y_1)$, into the gradient–intercept form of the straight line equation $y - y_1 = m(x - x_1)$ and determine the equation of the unknown line.

Worked example 13: Perpendicular lines

QUESTION

Determine the equation of the straight line passing through the point $T(2; 2)$ and perpendicular to the line $3y + 2x - 6 = 0$.

SOLUTION

Step 1: Write the equation in standard form

Let the gradient of the unknown line be m_1 and the given gradient be m_2. We write the given equation in gradient–intercept form and determine the value of m_2.

$$\begin{aligned} 3y + 2x - 6 &= 0 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2 \\ \therefore m_2 &= -\frac{2}{3} \end{aligned}$$

We know that the two lines are perpendicular, therefore $m_1 \times m_2 = -1$. Therefore $m_1 = \frac{3}{2}$.

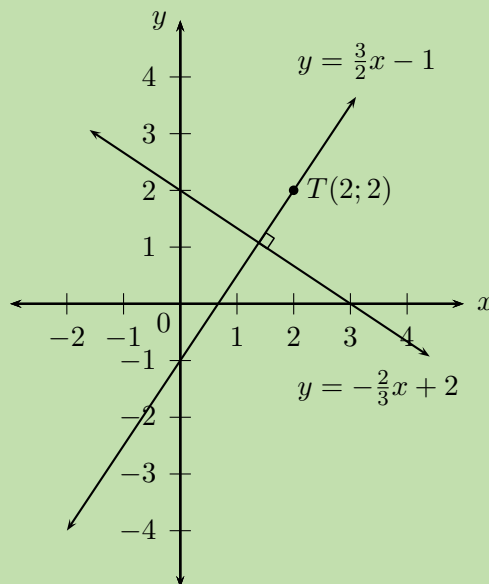
Step 2: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute $m_1 = \frac{3}{2}$.

$$y - y_1 = \frac{3}{2}(x - x_1)$$

Substitute the given point $T(2; 2)$.

$$\begin{aligned} y - 2 &= \frac{3}{2}(x - 2) \\ y - 2 &= \frac{3}{2}x - 3 \\ y &= \frac{3}{2}x - 1 \end{aligned}$$

A sketch was not required, but it is useful for checking the answer.

Step 3: Write the final answer

The equation of the straight line is $y = \frac{3}{2}x - 1$.

Worked example 14: Perpendicular lines

QUESTION

Determine the equation of the straight line passing through the point $(2; \frac{1}{3})$ and perpendicular to the line with an angle of inclination of 71,57°.

SOLUTION

Step 1: Use the given angle of inclination to determine gradient

Let the gradient of the unknown line be m_1 and let the given gradient be m_2.

$$\begin{aligned} m_2 &= \tan\theta \\ &= \tan 71{,}57^\circ \\ &= 3{,}0 \end{aligned}$$

Step 2: Determine the unknown gradient

Since we are given that the two lines are perpendicular,

$$\begin{aligned} m_1 \times m_2 &= -1 \\ \therefore m_1 &= -\frac{1}{3} \end{aligned}$$

Step 3: Write down the gradient–point form of the straight line equation

$$y - y_1 = m(x - x_1)$$

Substitute the gradient $m_1 = -\frac{1}{3}$.

$$y - y_1 = -\frac{1}{3}(x - x_1)$$

Substitute the given point $(2; \frac{1}{3})$.

$$\begin{aligned} y - \left(\frac{1}{3}\right) &= -\frac{1}{3}(x - 2) \\ y - \frac{1}{3} &= -\frac{1}{3}x + \frac{2}{3} \\ y &= -\frac{1}{3}x + 1 \end{aligned}$$

Step 4: Write the final answer

The equation of the straight line is $y = -\frac{1}{3}x + 1$.

See video: 22NG at www.everythingmaths.co.za

Exercise 4 – 8: Perpendicular lines

1. Calculate whether or not the following two lines are perpendicular:
   a) $y - 1 = 4x$ and $4y + x + 2 = 0$
   b) $10x = 5y - 1$ and $5y - x - 10 = 0$
   c) $x = y - 5$ and the line passing through $(-1; \frac{5}{4})$ and $(3; -\frac{11}{4})$
   d) $y = 2$ and $x = 1$
   e) $\frac{y}{3} = x$ and $3y + x = 9$
   f) $1 - 2x = y$ and the line passing through $(2; -1)$ and $(-1; 5)$
   g) $y = x + 2$ and $2y + 1 = 2x$
2. Determine the equation of the straight line that passes through the point $(-2; -4)$ and is perpendicular to the line $y + 2x = 1$.
3. Determine the equation of the straight line that passes through the point $(2; -7)$ and is perpendicular to the line $5y - x = 0$.

4. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with angle of inclination $\theta = 135°$.

5. Determine the equation of the straight line that passes through the point $(-2; \frac{2}{5})$ and is perpendicular to the line $y = \frac{4}{3}$.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

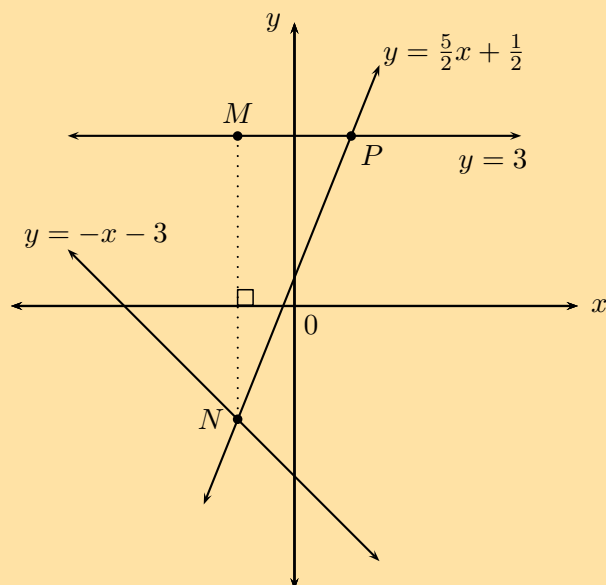
1a. 22NH 1b. 22NJ 1c. 22NK 1d. 22NM 1e. 22NN 1f. 22NP
1g. 22NQ 2. 22NR 3. 22NS 4. 22NT 5. 22NV

4.6 Summary EMBGH

See presentation: 22NW at www.everythingmaths.co.za

- Distance between two points: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Gradient of a line between two points: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- Mid-point of a line: $M(x; y) = \left(\dfrac{x_1 + x_2}{2}; \dfrac{y_1 + y_2}{2}\right)$
- Parallel lines: $m_1 = m_2$
- Perpendicular lines: $m_1 \times m_2 = -1$
- General form of a straight line equation: $ax + by + c = 0$
- Two-point form of a straight line equation: $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- Gradient–point form of a straight line equation: $y - y_1 = m(x - x_1)$
- Gradient–intercept form of a straight line equation (standard form): $y = mx + c$
- Angle of inclination of a straight line: θ, the angle formed between the line and the positive x-axis; $m = \tan\theta$

Exercise 4 – 9: End of chapter exercises

1. Determine the equation of the line:
   a) through points $(-1;3)$ and $(1;4)$
   b) through points $(7;-3)$ and $(0;4)$
   c) parallel to $y = \frac{1}{2}x + 3$ and passing through $(-2;3)$
   d) perpendicular to $y = -\frac{1}{2}x + 3$ and passing through $(-1;2)$
   e) perpendicular to $3y + x = 6$ and passing through the origin
2. Determine the angle of inclination of the following lines:
   a) $y = 2x - 3$
   b) $y = \frac{1}{3}x - 7$
   c) $4y = 3x + 8$
   d) $y = -\frac{2}{3}x + 3$
   e) $3y + x - 3 = 0$
3. $P(2;3)$, $Q(-4;0)$ and $R(5;-3)$ are the vertices of $\triangle PQR$ in the Cartesian plane. PR intersects the x-axis at S. Determine the following:
   a) the equation of the line PR
   b) the coordinates of point S
   c) the angle of inclination of PR (correct to two decimal places)
   d) the gradient of line PQ
   e) $Q\hat{P}R$
   f) the equation of the line perpendicular to PQ and passing through the origin
   g) the mid-point M of QR
   h) the equation of the line parallel to PR and passing through point M
4. Points $A(-3;5)$, $B(-7;-4)$ and $C(2;0)$ are given.
   a) Plot the points on the Cartesian plane.
   b) Determine the coordinates of D if $ABCD$ is a parallelogram.
   c) Prove that $ABCD$ is a rhombus.
5.

Consider the sketch above, with the following lines shown:

$y = -x - 3$

$y = 3$

$y = \frac{5}{2}x + \frac{1}{2}$

a) Determine the coordinates of the point N.

b) Determine the coordinates of the point P.

c) Determine the equation of the vertical line MN.

d) Determine the length of the vertical line MN.

e) Find $M\hat{N}P$.

f) Determine the equation of the line parallel to NP and passing through the point M.

6. The following points are given: $A(-2;3)$, $B(2;4)$, $C(3;0)$.

   a) Plot the points on the Cartesian plane.

   b) Prove that $\triangle ABC$ is a right-angled isosceles triangle.

   c) Determine the equation of the line AB.

   d) Determine the coordinates of D if $ABCD$ is a square.

   e) Determine the coordinates of E, the mid-point of BC.

7. Given points $S(2;5)$, $T(-3;-4)$ and $V(4;-2)$.

   a) Determine the equation of the line ST.

   b) Determine the size of $T\hat{S}V$.

8. Consider triangle FGH with vertices $F(-1;3)$, $G(2;1)$ and $H(4;4)$.

   a) Sketch $\triangle FGH$ on the Cartesian plane.

   b) Show that $\triangle FGH$ is an isosceles triangle.

   c) Determine the equation of the line PQ, perpendicular bisector of FH.

   d) Does G lie on the line PQ?

   e) Determine the equation of the line parallel to GH and passing through point F.

9. Given the points $A(-1;5)$, $B(5;-3)$ and $C(0;-6)$. M is the mid-point of AB and N is the mid-point of AC.

   a) Draw a sketch on the Cartesian plane.

   b) Show that the coordinates of M and N are $(2;1)$ and $(-\frac{1}{2};-\frac{1}{2})$ respectively.

   c) Use analytical geometry methods to prove the mid-point theorem. (Prove that $NM \parallel CB$ and $NM = \frac{1}{2}CB$.)

Think you got it? Get this answer and more practice on our Intelligent Practice Service

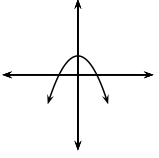
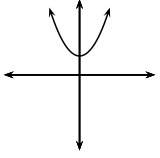
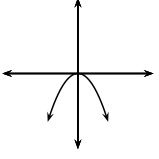
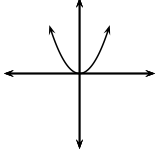
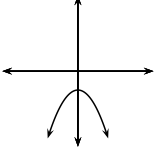
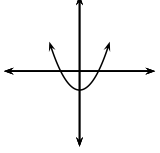
1a. 22NX 1b. 22NY 1c. 22NZ 1d. 22P2 1e. 22P3 2a. 22P4
2b. 22P5 2c. 22P6 2d. 22P7 2e. 22P8 3. 22P9 4. 22PB
5. 22PC 6. 22PD 7. 22PF 8. 22PG 9. 22PH

CHAPTER

Functions

A function describes a specific relationship between two variables; where an independent (input) variable has exactly one dependent (output) variable. Every element in the domain maps to only one element in the range. Functions can be one-to-one relations or many-to-one relations. A many-to-one relation associates two or more values of the independent variable with a single value of the dependent variable. Functions allow us to visualise relationships in the form of graphs, which are much easier to read and interpret than lists of numbers.

5.1 Quadratic functions EMBGJ

Revision EMBGK

Functions of the form $y = ax^2 + q$

Functions of the general form $y = ax^2 + q$ are called parabolic functions, where a and q are constants.

The effects of a and q on $f(x) = ax^2 + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(x)$ is shifted vertically upwards by q units.
    The turning point of $f(x)$ is above the x-axis.
  - For $q < 0$, $f(x)$ is shifted vertically downwards by q units.
    The turning point of $f(x)$ is below the x-axis.
  - q is also the y-intercept of the parabola.

- **The effect of a on shape**
  - For $a > 0$; the graph of $f(x)$ is a "smile" and has a minimum turning point $(0; q)$. As the value of a becomes larger, the graph becomes narrower.
    As a gets closer to 0, $f(x)$ becomes wider.
  - For $a < 0$; the graph of $f(x)$ is a "frown" and has a maximum turning point $(0; q)$. As the value of a becomes smaller, the graph becomes narrower.
    As a gets closer to 0, $f(x)$ becomes wider.

Exercise 5 – 1: Revision

1. On separate axes, accurately draw each of the following functions.
   - Use tables of values if necessary.
   - Use graph paper if available.

   a) $y_1 = x^2$
   b) $y_2 = \frac{1}{2}x^2$
   c) $y_3 = -x^2 - 1$
   d) $y_4 = -2x^2 + 4$

2. Use your sketches of the functions given above to complete the following table (the first column has been completed as an example):

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	$y_{\text{int}} = 0$			
value of a	$a = 1$			
effect of a	standard parabola			
turning point	$(0; 0)$			
axis of symmetry	$x = 0$ (y-axis)			
domain	$\{x : x \in \mathbb{R}\}$			
range	$\{y : y \geq 0\}$			

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22PJ 1b. 22PK 1c. 22PM 1d. 22PN 2. 22PP

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See video: 22PQ at www.everythingmaths.co.za

Functions of the form $y = a(x+p)^2 + q$ EMBGM

We now consider parabolic functions of the form $y = a(x+p)^2 + q$ and the effects of parameter p.

Investigation: The effects of a, p and q on a parabolic graph

1. On the same system of axes, plot the following graphs:

   a) $y_1 = x^2$
   b) $y_2 = (x-2)^2$
   c) $y_3 = (x-1)^2$
   d) $y_4 = (x+1)^2$
   e) $y_5 = (x+2)^2$

   Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
x-intercept(s)					
y-intercept					
turning point					
axis of symmetry					
domain					
range					
effect of p					

2. On the same system of axes, plot the following graphs:

   a) $y_1 = x^2 + 2$
   b) $y_2 = (x-2)^2 - 1$
   c) $y_3 = (x-1)^2 + 1$
   d) $y_4 = (x+1)^2 + 1$
   e) $y_5 = (x+2)^2 - 1$

   Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
x-intercept(s)					
y-intercept					
turning point					
axis of symmetry					
domain					
range					
effect of q					

3. Consider the three functions given below and answer the questions that follow:

   - $y_1 = (x-2)^2 + 1$
   - $y_2 = 2(x-2)^2 + 1$
   - $y_3 = -\frac{1}{2}(x-2)^2 + 1$

   a) What is the value of a for y_2?
   b) Does y_1 have a minimum or maximum turning point?

c) What are the coordinates of the turning point of y_2?

d) Compare the graphs of y_1 and y_2. Discuss the similarities and differences.

e) What is the value of a for y_3?

f) Will the graph of y_3 be narrower or wider than the graph of y_1?

g) Determine the coordinates of the turning point of y_3.

h) Compare the graphs of y_1 and y_3. Describe any differences.

See video: 22PR at www.everythingmaths.co.za

The effect of the parameters on $y = a(x + p)^2 + q$

The effect of p is a horizontal shift because all points are moved the same distance in the same direction (the entire graph slides to the left or to the right).

- For $p > 0$, the graph is shifted to the left by p units.
- For $p < 0$, the graph is shifted to the right by p units.

The value of p also affects whether the turning point is to the left of the y-axis $(p > 0)$ or to the right of the y-axis $(p < 0)$. The axis of symmetry is the line $x = -p$.

The effect of q is a vertical shift. The value of q affects whether the turning point of the graph is above the x-axis $(q > 0)$ or below the x-axis $(q < 0)$.

The value of a affects the shape of the graph. If $a < 0$, the graph is a "frown" and has a maximum turning point. If $a > 0$ then the graph is a "smile" and has a minimum turning point. When $a = 0$, the graph is a horizontal line $y = q$.

	$p > 0$		$p < 0$	
	$a < 0$	$a > 0$	$a < 0$	$a > 0$
$q > 0$				
$q < 0$				

See simulation: 22PS at www.everythingmaths.co.za

Discovering the characteristics

For functions of the general form $f(x) = y = a(x+p)^2 + q$:

Domain and range

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $f(x)$ is undefined.

The range of $f(x)$ depends on whether the value for a is positive or negative. If $a > 0$ we have:

$$\begin{aligned} (x+p)^2 &\geq 0 \quad \text{(perfect square is always positive)} \\ \therefore a(x+p)^2 &\geq 0 \quad (a \text{ is positive}) \\ \therefore a(x+p)^2 + q &\geq q \\ \therefore f(x) &\geq q \end{aligned}$$

The range is therefore $\{y : y \geq q, y \in \mathbb{R}\}$ if $a > 0$. Similarly, if $a < 0$, the range is $\{y : y \leq q, y \in \mathbb{R}\}$.

Worked example 1: Domain and range

QUESTION

State the domain and range for $g(x) = -2(x-1)^2 + 3$.

SOLUTION

Step 1: Determine the domain

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $g(x)$ is undefined.

Step 2: Determine the range

The range of $g(x)$ can be calculated from:

$$\begin{aligned} (x-1)^2 &\geq 0 \\ -2(x-1)^2 &\leq 0 \\ -2(x-1)^2 + 3 &\leq 3 \\ g(x) &\leq 3 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \leq 3\}$ or in interval notation $(-\infty; 3]$.

Notice in the example above that it helps to have the function in the form $y = a(x+p)^2 + q$.

We use the method of **completing the square** to write a quadratic function of the general form $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$ (see Chapter 2).

Exercise 5 – 2: Domain and range

Give the domain and range for each of the following functions:

1. $f(x) = (x-4)^2 - 1$
2. $g(x) = -(x-5)^2 + 4$
3. $h(x) = x^2 - 6x + 9$
4. $j(x) = -2(x+1)^2$
5. $k(x) = -x^2 + 2x - 3$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22PT 2. 22PV 3. 22PW 4. 22PX 5. 22PY

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Intercepts

The y-intercept:

Every point on the y-axis has an x-coordinate of 0, therefore to calculate the y-intercept we let $x = 0$.

For example, the y-intercept of $g(x) = (x-1)^2 + 5$ is determined by setting $x = 0$:

$$\begin{aligned} g(x) &= (x-1)^2 + 5 \\ g(0) &= (0-1)^2 + 5 \\ &= 6 \end{aligned}$$

This gives the point $(0;6)$.

The x-intercept:

Every point on the x-axis has a y-coordinate of 0, therefore to calculate the x-intercept we let $y = 0$.

For example, the x-intercept of $g(x) = (x-1)^2 + 5$ is determined by setting $y = 0$:

$$\begin{aligned} g(x) &= (x-1)^2 + 5 \\ 0 &= (x-1)^2 + 5 \\ -5 &= (x-1)^2 \end{aligned}$$

which has no real solutions. Therefore, the graph of $g(x)$ lies above the x-axis and does not have any x-intercepts.

Exercise 5 – 3: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x) = (x+4)^2 - 1$
2. $g(x) = 16 - 8x + x^2$
3. $h(x) = -x^2 + 4x - 3$
4. $j(x) = 4(x-3)^2 - 1$
5. $k(x) = 4(x-3)^2 + 1$
6. $l(x) = 2x^2 - 3x - 4$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22PZ 2. 22Q2 3. 22Q3 4. 22Q4 5. 22Q5 6. 22Q6

Turning point

The turning point of the function $f(x) = a(x+p)^2 + q$ is determined by examining the range of the function:

- If $a > 0$, $f(x)$ has a minimum turning point and the range is $[q; \infty)$:
  The minimum value of $f(x)$ is q.
  If $f(x) = q$, then $a(x+p)^2 = 0$, and therefore $x = -p$.
  This gives the turning point $(-p; q)$.
- If $a < 0$, $f(x)$ has a maximum turning point and the range is $(-\infty; q]$:
  The maximum value of $f(x)$ is q.
  If $f(x) = q$, then $a(x+p)^2 = 0$, and therefore $x = -p$.
  This gives the turning point $(-p; q)$.

Therefore the turning point of the quadratic function $f(x) = a(x+p)^2 + q$ is $(-p; q)$.

Alternative form for quadratic equations:

We can also write the quadratic equation in the form

$$y = a(x-p)^2 + q$$

The effect of p is still a horizontal shift, however notice that:

- For $p > 0$, the graph is shifted to the **right** by p units.
- For $p < 0$, the graph is shifted to the **left** by p units.

The turning point is $(p; q)$ and the axis of symmetry is the line $x = p$.

Worked example 2: Turning point

QUESTION

Determine the turning point of $g(x) = 3x^2 - 6x - 1$.

SOLUTION

Step 1: Write the equation in the form $y = a(x+p)^2 + q$

We use the method of completing the square:

$$\begin{aligned} g(x) &= 3x^2 - 6x - 1 \\ &= 3(x^2 - 2x) - 1 \\ &= 3\left((x-1)^2 - 1\right) - 1 \\ &= 3(x-1)^2 - 3 - 1 \\ &= 3(x-1)^2 - 4 \end{aligned}$$

Step 2: Determine turning point $(-p; q)$

From the equation $g(x) = 3(x-1)^2 - 4$ we know that the turning point for $g(x)$ is $(1; -4)$.

Worked example 3: Turning point

QUESTION

1. Show that the x-value for the turning point of $h(x) = ax^2 + bx + c$ is given by $x = -\frac{b}{2a}$.
2. Hence, determine the turning point of $k(x) = 2 - 10x + 5x^2$.

SOLUTION

Step 1: Write the equation in the form $y = a(x+p)^2 + q$ and show that $p = \frac{b}{2a}$

We use the method of completing the square:

$$\begin{aligned} h(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \end{aligned}$$

Take half the coefficient of the x term and square it; then add and subtract it from the

expression.

$$\begin{aligned}
h(x) &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

From the above we have that the turning point is at $x = -p = -\frac{b}{2a}$ and $y = q = -\frac{b^2-4ac}{4a}$.

Step 2: Determine the turning point of $k(x)$

Write the equation in the general form $y = ax^2 + bx + c$.

$$k(x) = 5x^2 - 10x + 2$$

Therefore $a = 5$; $b = -10$; $c = 2$.

Use the results obtained above to determine $x = -\frac{b}{2a}$:

$$\begin{aligned}
x &= -\left(\frac{-10}{2(5)}\right) \\
&= 1
\end{aligned}$$

Substitute $x = 1$ to obtain the corresponding y-value :

$$\begin{aligned}
y &= 5x^2 - 10x + 2 \\
&= 5(1)^2 - 10(1) + 2 \\
&= 5 - 10 + 2 \\
&= -3
\end{aligned}$$

The turning point of $k(x)$ is $(1; -3)$.

Exercise 5 – 4: Turning points

Determine the turning point of each of the following:

1. $y = x^2 - 6x + 8$
2. $y = -x^2 + 4x - 3$
3. $y = \frac{1}{2}(x + 2)^2 - 1$
4. $y = 2x^2 + 2x + 1$

5. $y = 18 + 6x - 3x^2$
6. $y = -2[(x + 1)^2 + 3]$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22Q7 2. 22Q8 3. 22Q9 4. 22QB 5. 22QC 6. 22QD

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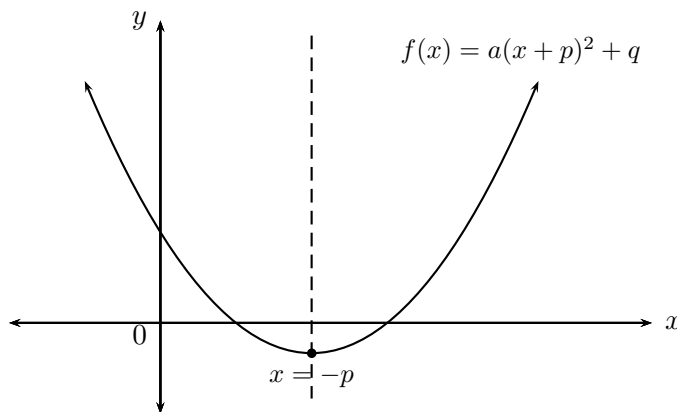
Axis of symmetry

The axis of symmetry for $f(x) = a(x + p)^2 + q$ is the vertical line $x = -p$. The axis of symmetry passes through the turning point $(-p; q)$ and is parallel to the y-axis.

Exercise 5 – 5: Axis of symmetry

1. Determine the axis of symmetry of each of the following:
   a) $y = 2x^2 - 5x - 18$
   b) $y = 3(x - 2)^2 + 1$
   c) $y = 4x - x^2$
2. Write down the equation of a parabola where the y-axis is the axis of symmetry.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22QF 1b. 22QG 1c. 22QH 2. 22QJ

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Sketching graphs of the form $f(x) = a(x+p)^2 + q$

In order to sketch graphs of the form $f(x) = a(x+p)^2 + q$, we need to determine five characteristics:

- sign of a
- turning point
- y-intercept
- x-intercept(s) (if they exist)
- domain and range

See video: 22QK at www.everythingmaths.co.za

Worked example 4: Sketching a parabola

QUESTION

Sketch the graph of $y = -\frac{1}{2}(x+1)^2 - 3$.

Mark the intercepts, turning point and the axis of symmetry. State the domain and range of the function.

SOLUTION

Step 1: Examine the equation of the form $y = a(x+p)^2 + q$

We notice that $a < 0$, therefore the graph is a "frown" and has a maximum turning point.

Step 2: Determine the turning point $(-p; q)$

From the equation we know that the turning point is $(-1; -3)$.

Step 3: Determine the axis of symmetry $x = -p$

From the equation we know that the axis of symmetry is $x = -1$.

Step 4: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= -\frac{1}{2}\left((0)+1\right)^2 - 3 \\ &= -\frac{1}{2} - 3 \\ &= -3\frac{1}{2} \end{aligned}$$

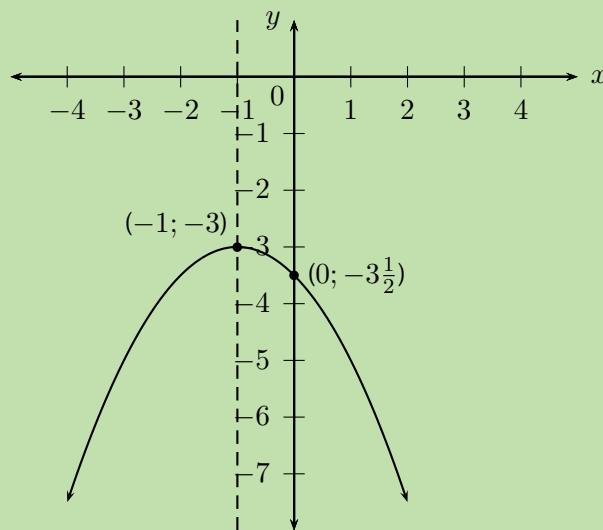
This gives the point $(0; -3\frac{1}{2})$.

Step 5: Determine the x-intercepts

The x-intercepts are obtained by letting $y = 0$:

$$\begin{aligned} 0 &= -\frac{1}{2}(x+1)^2 - 3 \\ 3 &= -\frac{1}{2}(x+1)^2 \\ -6 &= (x+1)^2 \end{aligned}$$

which has no real solutions. Therefore, there are no x-intercepts and the graph lies below the x-axis.

Step 6: Plot the points and sketch the graph

Step 7: State the domain and range

Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y \leq -3, y \in \mathbb{R}\}$

See video: 22QM at www.everythingmaths.co.za

Worked example 5: Sketching a parabola

QUESTION

Sketch the graph of $y = \frac{1}{2}x^2 - 4x + \frac{7}{2}$.

Determine the intercepts, turning point and the axis of symmetry. Give the domain and range of the function.

SOLUTION

Step 1: Examine the equation of the form $y = ax^2 + bx + c$

We notice that $a > 0$, therefore the graph is a "smile" and has a minimum turning point.

Step 2: Determine the turning point and the axis of symmetry

Check that the equation is in standard form and identify the coefficients.

$$a = \frac{1}{2}; \qquad b = -4; \qquad c = \frac{7}{2}$$

Calculate the x-value of the turning point using

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\left(\frac{-4}{2\left(\frac{1}{2}\right)}\right) \\ &= 4 \end{aligned}$$

Therefore the axis of symmetry is $x = 4$.

Substitute $x = 4$ into the original equation to obtain the corresponding y-value.

$$\begin{aligned} y &= \frac{1}{2}x^2 - 4x + \frac{7}{2} \\ &= \frac{1}{2}(4)^2 - 4(4) + \frac{7}{2} \\ &= 8 - 16 + \frac{7}{2} \\ &= -4\frac{1}{2} \end{aligned}$$

This gives the point $\left(4; -4\frac{1}{2}\right)$.

Step 3: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= \frac{1}{2}(0)^2 - 4(0) + \frac{7}{2} \\ &= \frac{7}{2} \end{aligned}$$

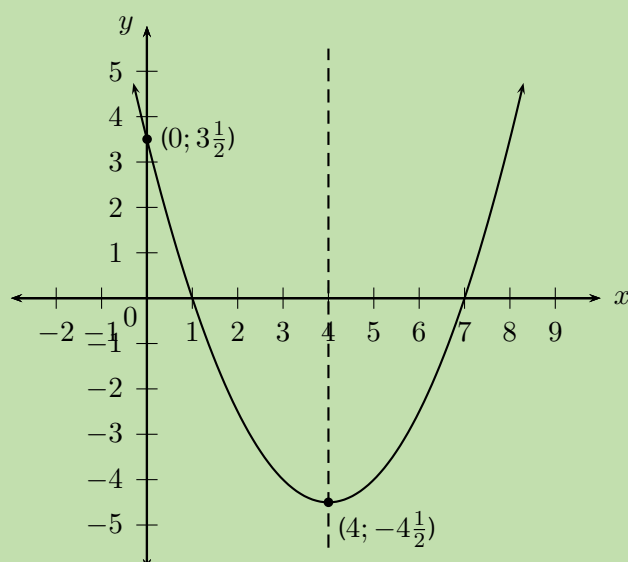
This gives the point $\left(0; \frac{7}{2}\right)$.

Step 4: Determine the x-intercepts

The x-intercepts are obtained by letting $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{2}x^2 - 4x + \frac{7}{2} \\ &= x^2 - 8x + 7 \\ &= (x - 1)(x - 7) \end{aligned}$$

Therefore $x = 1$ or $x = 7$. This gives the points $(1; 0)$ and $(7; 0)$.

Step 5: Plot the points and sketch the graph

Step 6: State the domain and range

Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y \geq -4\frac{1}{2}, y \in \mathbb{R}\}$

See video: 22QN at www.everythingmaths.co.za

Investigation: Shifting the equation of a parabola

Carl and Eric are doing their Mathematics homework and decide to check each others answers.

Homework question:

If the parabola $y = 3x^2 + 1$ is shifted 2 units to the right, determine the equation of the new parabola.

- Carl's answer:

  A shift to the right means moving in the positive x direction, therefore x is replaced with $x + 2$ and the new equation is $y = 3(x + 2)^2 + 1$.

- Eric's answer:

  We replace x with $x - 2$, therefore the new equation is $y = 3(x - 2)^2 + 1$.

Work together in pairs. Discuss the two different answers and decide which one is correct. Use calculations and sketches to help explain your reasoning.

Writing an equation of a shifted parabola

The parabola is shifted horizontally:

- If the parabola is shifted m units to the right, x is replaced by $(x - m)$.
- If the parabola is shifted m units to the left, x is replaced by $(x + m)$.

The parabola is shifted vertically:

- If the parabola is shifted n units down, y is replaced by $(y + n)$.
- If the parabola is shifted n units up, y is replaced by $(y - n)$.

Worked example 6: Shifting a parabola

QUESTION

Given $y = x^2 - 2x - 3$.

1. If the parabola is shifted 1 unit to the right, determine the new equation of the parabola.
2. If the parabola is shifted 3 units down, determine the new equation of the parabola.

SOLUTION

Step 1: Determine the new equation of the shifted parabola

1. The parabola is shifted 1 unit to the right, so x must be replaced by $(x - 1)$.

$$\begin{aligned} y &= x^2 - 2x - 3 \\ &= (x-1)^2 - 2(x-1) - 3 \\ &= x^2 - 2x + 1 - 2x + 2 - 3 \\ &= x^2 - 4x \end{aligned}$$

   Be careful not to make a common error: replacing x with $x + 1$ for a shift to the right.

2. The parabola is shifted 3 units down, so y must be replaced by $(y + 3)$.

$$\begin{aligned} y + 3 &= x^2 - 2x - 3 \\ y &= x^2 - 2x - 3 - 3 \\ &= x^2 - 2x - 6 \end{aligned}$$

Exercise 5 – 6: Sketching parabolas

1. Sketch graphs of the following functions and determine:

   - intercepts
   - turning point
   - axes of symmetry
   - domain and range

   a) $y = -x^2 + 4x + 5$
   b) $y = 2(x + 1)^2$
   c) $y = 3x^2 - 2(x + 2)$
   d) $y = 3(x - 2)^2 + 1$

2. Draw the following graphs on the same system of axes:
   $f(x) = -x^2 + 7$
   $g(x) = -(x - 2)^2 + 7$
   $h(x) = (x - 2)^2 - 7$

3. Draw a sketch of each of the following graphs:

   a) $y = ax^2 + bx + c$ if $a > 0$, $b > 0$, $c < 0$.
   b) $y = ax^2 + bx + c$ if $a < 0$, $b = 0$, $c > 0$.
   c) $y = ax^2 + bx + c$ if $a < 0$, $b < 0$, $b^2 - 4ac < 0$.
   d) $y = (x + p)^2 + q$ if $p < 0$, $q < 0$ and the x-intercepts have different signs.
   e) $y = a(x + p)^2 + q$ if $a < 0$, $p < 0$, $q > 0$ and one root is zero.
   f) $y = a(x + p)^2 + q$ if $a > 0$, $p = 0$, $b^2 - 4ac > 0$.

4. Determine the new equation (in the form $y = ax^2 + bx + c$) if:

   a) $y = 2x^2 + 4x + 2$ is shifted 3 units to the left.
   b) $y = -(x + 1)^2$ is shifted 1 unit up.
   c) $y = 3(x - 1)^2 + 2\left(x - \frac{1}{2}\right)$ is shifted 2 units to the right.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22QP 1b. 22QQ 1c. 22QR 1d. 22QS 2. 22QT 3a. 22QV
3b. 22QW 3c. 22QX 3d. 22QY 3e. 22QZ 3f. 22R2 4a. 22R3
4b. 22R4 4c. 22R5

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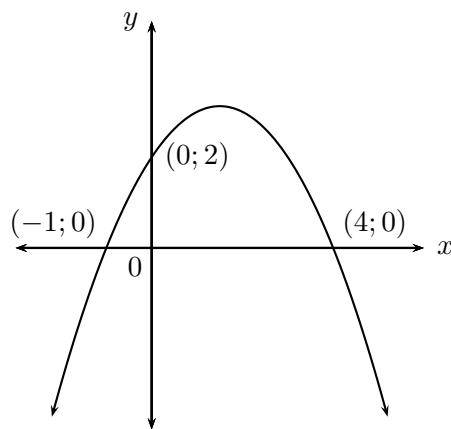
Finding the equation of a parabola from the graph

If the intercepts are given, use $y = a(x - x_1)(x - x_2)$.

Example:

x-intercepts: $(-1;0)$ and $(4;0)$

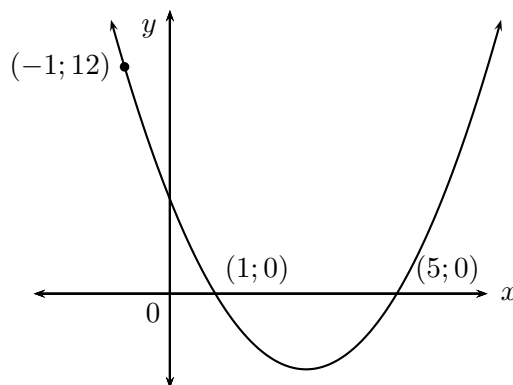
$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= a(x+1)(x-4) \\ &= ax^2 - 3ax - 4a \end{aligned}$$

y-intercept: $(0;2)$

$$\begin{aligned} -4a &= 2 \\ a &= -\frac{1}{2} \end{aligned}$$

Equation of the parabola:

$$\begin{aligned} y &= ax^2 - 3ax - 4a \\ &= -\frac{1}{2}x^2 - 3\left(-\frac{1}{2}\right)x - 4\left(-\frac{1}{2}\right) \\ &= -\frac{1}{2}x^2 + \frac{3}{2}x + 2 \end{aligned}$$

If the x-intercepts and another point are given, use $y = a(x - x_1)(x - x_2)$.

Example:

x-intercepts: $(1;0)$ and $(5;0)$

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= a(x-1)(x-5) \\ &= ax^2 - 6ax + 5a \end{aligned}$$

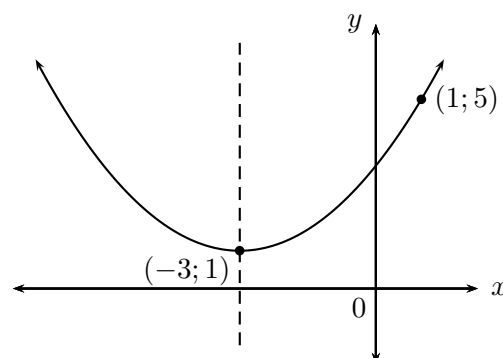
Substitute the point: $(-1;12)$

$$\begin{aligned} 12 &= a(-1)^2 - 6a(-1) + 5a \\ 12 &= a + 6a + 5a \\ 12 &= 12a \\ 1 &= a \end{aligned}$$

Equation of the parabola:

$$\begin{aligned} y &= ax^2 - 6ax + 5a \\ &= x^2 - 6x + 5 \end{aligned}$$

If the turning point and another point are given, use $y = a(x+p)^2 + q$.

Example:

Turning point: $(-3;1)$

$$\begin{aligned} y &= a(x+p)^2 + q \\ &= a(x+3)^2 + 1 \\ &= ax^2 + 6ax + 9a + 1 \end{aligned}$$

Substitute the point: $(1;5)$

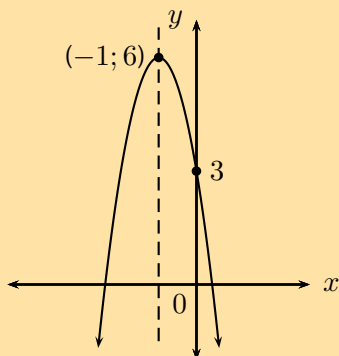
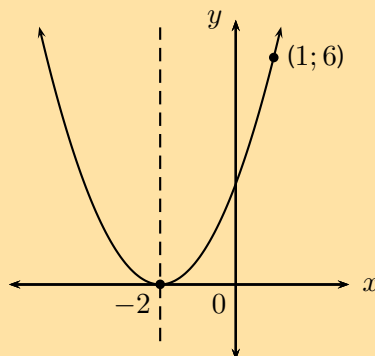
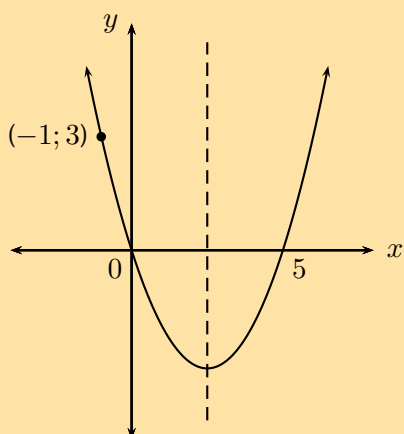
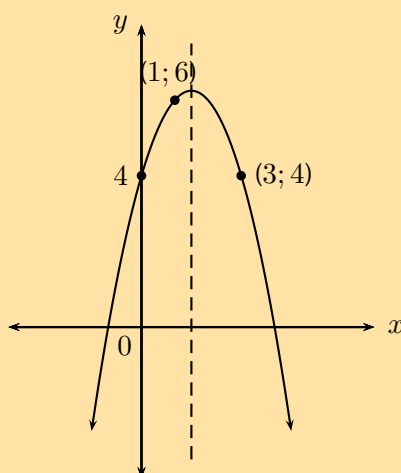
$$\begin{aligned} 5 &= a(1)^2 + 6a(1) + 9a + 1 \\ 4 &= 16a \\ \frac{1}{4} &= a \end{aligned}$$

Equation of the parabola:

$y = \frac{1}{4}(x+3)^2 + 1$

Exercise 5 – 7: Finding the equation

Determine the equations of the following graphs. Write your answers in the form $y = a(x + p)^2 + q$.

1.

2.

3.

4.

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1. 22R6 2. 22R7 3. 22R8 4. 22R9

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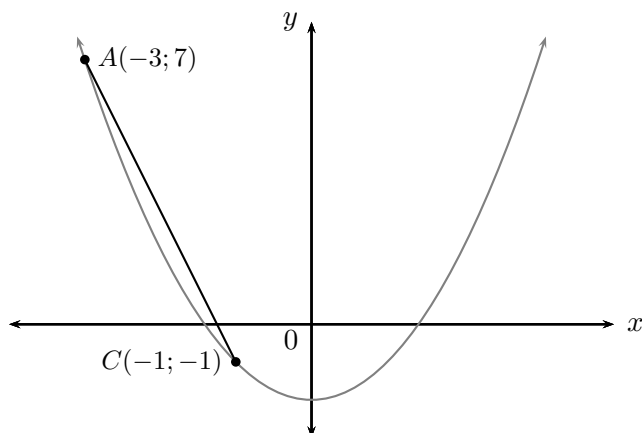
5.2 Average gradient EMBGN

We notice that the gradient of a curve changes at every point on the curve, therefore we need to work with the average gradient. The average gradient between any two points on a curve is the gradient of the straight line passing through the two points.

For the diagram above, the gradient of the line AC is

$$\begin{aligned} \text{Gradient} &= \frac{y_A - y_C}{x_A - x_C} \\ &= \frac{7 - (-1)}{-3 - (-1)} \\ &= \frac{8}{-2} \\ &= -4 \end{aligned}$$

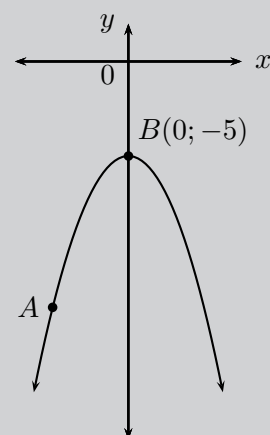
This is the average gradient of the curve between the points A and C.

What happens to the gradient if we fix the position of one point and move the second point closer to the fixed point?

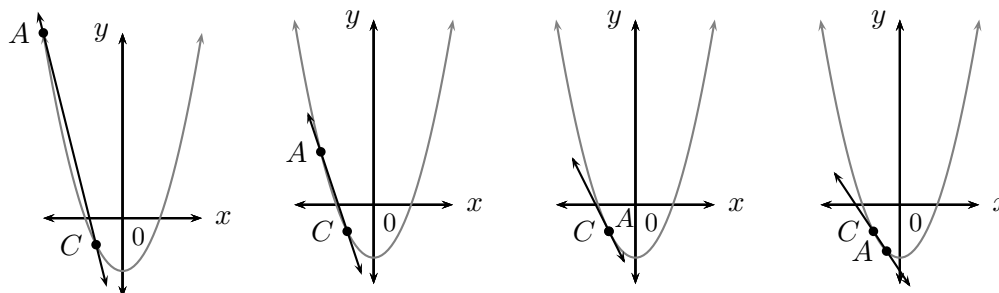
See video: 22RB at www.everythingmaths.co.za

Investigation: Gradient at a single point on a curve

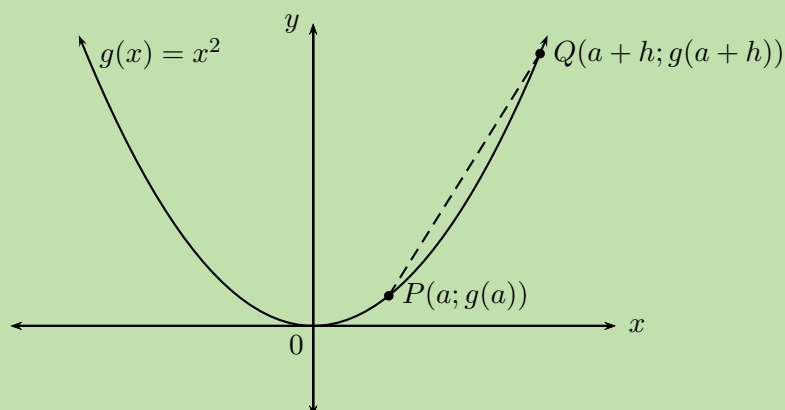
The curve shown here is defined by $y = -2x^2 - 5$. Point B is fixed at $(0; -5)$ and the position of point A varies.

Complete the table below by calculating the y-coordinates of point A for the given x-coordinates and then calculating the average gradient between points A and B.

x_A	y_A	Average gradient
-2		
$-1{,}5$		
-1		
$-0{,}5$		
0		
$0{,}5$		
1		
$1{,}5$		
2		

1. What happens to the average gradient as A moves towards B?
2. What happens to the average gradient as A moves away from B?
3. What is the average gradient when A overlaps with B?

In the example above, the gradient of the straight line that passes through points A and C changes as A moves closer to C. At the point where A and C overlap, the straight line only passes through one point on the curve. This line is known as a tangent to the curve.

We therefore introduce the idea of the gradient at a single point on a curve. The gradient at a point on a curve is the gradient of the tangent to the curve at the given point.

Worked example 7: Average gradient

QUESTION

1. Find the average gradient between two points $P\,(a; g(a))$ and $Q\,(a+h; g(a+h))$ on a curve $g(x) = x^2$.
2. Determine the average gradient between $P\,(2; g(2))$ and $Q\,(5; g(5))$.
3. Explain what happens to the average gradient if Q moves closer to P.

SOLUTION

Step 1: Assign labels to the x-values for the given points

$$\begin{aligned} x_1 &= a \\ x_2 &= a + h \end{aligned}$$

Step 2: Determine the corresponding y-coordinates

Using the function $g(x) = x^2$, we can determine:

$$\begin{aligned} y_1 &= g(a) \\ &= a^2 \end{aligned}$$

$$\begin{aligned} y_2 &= g(a+h) \\ &= (a+h)^2 \\ &= a^2 + 2ah + h^2 \end{aligned}$$

Step 3: Calculate the average gradient

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{\left(a^2 + 2ah + h^2\right) - \left(a^2\right)}{(a+h) - (a)} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{a + h - a} \\ &= \frac{2ah + h^2}{h} \\ &= \frac{h(2a+h)}{h} \\ &= 2a + h\end{aligned}$$

The average gradient between $P\left(a; g(a)\right)$ and $Q\left(a+h; g(a+h)\right)$ on the curve $g(x) = x^2$ is $2a + h$.

Step 4: Calculate the average gradient between $P\left(2; g(2)\right)$ and $Q\left(5; g(5)\right)$

The x-coordinate of P is a and the x-coordinate of Q is $a+h$ therefore if we know that $a = 2$ and $a + h = 5$, then $h = 3$.

The average gradient is therefore $2a + h = 2\left(2\right) + \left(3\right) = 7$

Step 5: When Q moves closer to P

When point Q moves closer to point P, h gets smaller.

When the point Q overlaps with the point P, $h = 0$ and the gradient is given by $2a$.

We can write the equation for average gradient in another form. Given a curve $f(x)$ with two points P and Q with $P\left(a; f(a)\right)$ and $Q\left(a+h; f(a+h)\right)$. The average gradient between P and Q is:

$$\begin{aligned}\text{Average gradient } &= \frac{y_Q - y_P}{x_Q - x_P} \\ &= \frac{f(a+h) - f(a)}{(a+h) - (a)} \\ &= \frac{f(a+h) - f(a)}{h}\end{aligned}$$

This result is important for calculating the gradient at a point on a curve and will be explored in greater detail in Grade 12.

Worked example 8: Average gradient

QUESTION

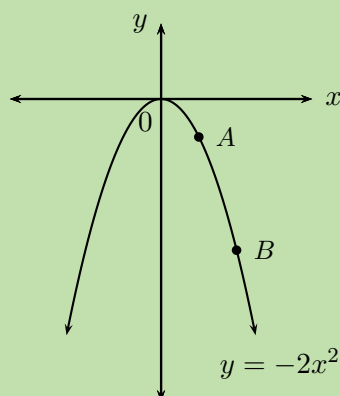
Given $f(x) = -2x^2$.

1. Draw a sketch of the function and determine the average gradient between the points A, where $x = 1$, and B, where $x = 3$.
2. Determine the gradient of the curve at point A.

SOLUTION

Step 1: Examine the form of the equation

From the equation we see that $a < 0$, therefore the graph is a "frown" and has a maximum turning point. We also see that when $x = 0$, $y = 0$, therefore the graph passes through the origin.

Step 2: Draw a rough sketch

Step 3: Calculate the average gradient between A and B

$$\begin{aligned}\text{Average gradient} &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{-2(3)^2 - (-2(1)^2)}{2} \\ &= \frac{-18 + 2}{2} \\ &= \frac{-16}{2} \\ &= -8\end{aligned}$$

Step 4: Calculate the average gradient for $f(x)$

$$\begin{aligned}\text{Average gradient} &= \frac{f(a+h)-f(a)}{(a+h)-a} \\ &= \frac{-2(a+h)^2-(-2a^2)}{h} \\ &= \frac{-2a^2-4ah-2h^2+2a^2}{h} \\ &= \frac{-4ah-2h^2}{h} \\ &= \frac{h(-4a-2h)}{h} \\ &= -4a-2h\end{aligned}$$

At point A, $h = 0$ and $a = 1$. Therefore

$$\begin{aligned}\text{Average gradient} &= -4a-2h \\ &= -4(1)-2(0) \\ &= -4\end{aligned}$$

Exercise 5 – 8:

1. a) Determine the average gradient of the curve $f(x) = x(x+3)$ between $x = 5$ and $x = 3$.
   b) Hence, state what you can deduce about the function f between $x = 5$ and $x = 3$.
2. $A(1;3)$ is a point on $f(x) = 3x^2$.
   a) Draw a sketch of $f(x)$ and label point A.
   b) Determine the gradient of the curve at point A.
   c) Determine the equation of the tangent line at A.
3. Given: $g(x) = -x^2 + 1$.
   a) Draw a sketch of $g(x)$.
   b) Determine the average gradient of the curve between $x = -2$ and $x = 1$.
   c) Determine the gradient of g at $x = 2$.
   d) Determine the gradient of g at $x = 0$.

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1. 22RC 2. 22RD 3. 22RF

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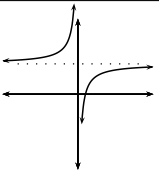
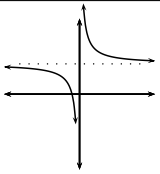
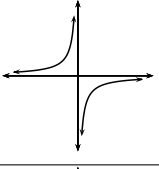
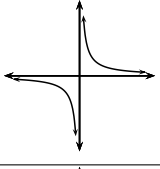
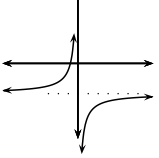
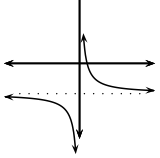
5.3 Hyperbolic functions EMBGP

Revision EMBGQ

Functions of the form $y = \frac{a}{x} + q$

Functions of the general form $y = \frac{a}{x} + q$ are called hyperbolic functions, where a and q are constants.

The effects of a and q on $f(x) = \frac{a}{x} + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(x)$ is shifted vertically upwards by q units.
  - For $q < 0$, $f(x)$ is shifted vertically downwards by q units.
  - The horizontal asymptote is the line $y = q$.
  - The vertical asymptote is the y-axis, the line $x = 0$.

- **The effect of a on shape and quadrants**
  - For $a > 0$, $f(x)$ lies in the first and third quadrants.
  - For $a > 1$, $f(x)$ will be further away from both axes than $y = \frac{1}{x}$.
  - For $0 < a < 1$, as a tends to 0, $f(x)$ moves closer to the axes than $y = \frac{1}{x}$.
  - For $a < 0$, $f(x)$ lies in the second and fourth quadrants.
  - For $a < -1$, $f(x)$ will be further away from both axes than $y = -\frac{1}{x}$.
  - For $-1 < a < 0$, as a tends to 0, $f(x)$ moves closer to the axes than $y = -\frac{1}{x}$.

Exercise 5 – 9: Revision

1. Consider the following hyperbolic functions:
   - $y_1 = \frac{1}{x}$
   - $y_2 = -\frac{4}{x}$
   - $y_3 = \frac{4}{x} - 2$
   - $y_4 = -\frac{4}{x} + 1$

Complete the table to summarise the properties of the hyperbolic function:

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	no vertical shift			
value of a	$a = 1$			
effect of a	lies in I and III quad			
asymptotes	y-axis, $x = 0$ x-axis, $y = 0$			
axes of symmetry	$y = x$ $y = -x$			
domain	$\{x : x \in \mathbb{R}, x \neq 0\}$			
range	$\{y : y \in \mathbb{R}, y \neq 0\}$			

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1. 22RG

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See video: 22RH at www.everythingmaths.co.za

Functions of the form $y = \frac{a}{x+p} + q$ EMBGR

We now consider hyperbolic functions of the form $y = \frac{a}{x+p} + q$ and the effects of parameter p.

Investigation: The effects of a, p and q on a hyperbolic graph

1. On the same system of axes, plot the following graphs:

   a) $y_1 = \frac{1}{x}$
   b) $y_2 = \frac{1}{x-2}$
   c) $y_3 = \frac{1}{x-1}$
   d) $y_4 = \frac{1}{x+1}$

   Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4
intercept(s)				
asymptotes				
axes of symmetry				
domain				
range				
effect of p				

2. Complete the following sentences for functions of the form $y = \frac{a}{x+p} + q$:

   a) A change in p causes a ...... shift.

   b) If the value of p increases, the graph and the vertical asymptote ......

   c) If the value of q changes, then the ...... asymptote of the hyperbola will shift.

   d) If the value of p decreases, the graph and the vertical asymptote ......

The effect of the parameters on $y = \frac{a}{x+p} + q$

The effect of p is a horizontal shift because all points are moved the same distance in the same direction (the entire graph slides to the left or to the right).

- For $p > 0$, the graph is shifted to the left by p units.
- For $p < 0$, the graph is shifted to the right by p units.

The value of p also affects the vertical asymptote, the line $x = -p$.

The effect of q is a vertical shift. The value of q also affects the horizontal asymptotes, the line $y = q$.

The value of a affects the shape of the graph and its position on the Cartesian plane.

Discovering the characteristics

For functions of the general form: $f(x) = y = \frac{a}{x+p} + q$:

Domain and range

The domain is $\{x : x \in \mathbb{R}, x \neq -p\}$. If $x = -p$, the dominator is equal to zero and the function is undefined.

We see that

$$y = \frac{a}{x+p} + q$$

can be re-written as:

$$y - q = \frac{a}{x+p}$$

If $x \neq -p$ then:

$$\begin{aligned}(y-q)(x+p) &= a \\ x + p &= \frac{a}{y-q}\end{aligned}$$

The range is therefore $\{y : y \in \mathbb{R}, y \neq q\}$.

These restrictions on the domain and range determine the vertical asymptote $x = -p$ and the horizontal asymptote $y = q$.

Worked example 9: Domain and range

QUESTION

Determine the domain and range for $g(x) = \frac{2}{x+1} + 2$.

SOLUTION

Step 1: Determine the domain

The domain is $\{x : x \in \mathbb{R}, x \neq -1\}$ since $g(x)$ is undefined for $x = -1$.

Step 2: Determine the range

Let $g(x) = y$:

$$\begin{aligned}y &= \frac{2}{x+1} + 2 \\ y - 2 &= \frac{2}{x+1} \\ (y-2)(x+1) &= 2 \\ x + 1 &= \frac{2}{y-2}\end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in \mathbb{R}, g(x) \neq 2\}$.

Exercise 5 – 10: Domain and range

Determine the domain and range for each of the following functions:

1. $y = \frac{1}{x} + 1$
2. $g(x) = \frac{8}{x-8} + 4$
3. $y = -\frac{4}{x+1} - 3$
4. $x = \frac{2}{3-y} + 5$
5. $(y-2)(x+2) = 3$

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1. 22RJ 2. 22RK 3. 22RM 4. 22RN 5. 22RP

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Intercepts

The y-intercept:

To calculate the y-intercept we let $x = 0$. For example, the y-intercept of $g(x) = \frac{2}{x+1} + 2$ is determined by setting $x = 0$:

$$\begin{aligned} g(x) &= \frac{2}{x+1} + 2 \\ g(0) &= \frac{2}{0+1} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

This gives the point $(0; 4)$.

The x-intercept:

To calculate the x-intercept we let $y = 0$. For example, the x-intercept of $g(x) = \frac{2}{x+1} + 2$ is determined by setting $y = 0$:

$$\begin{aligned} g(x) &= \frac{2}{x+1} + 2 \\ 0 &= \frac{2}{x+1} + 2 \\ -2 &= \frac{2}{x+1} \\ -2(x+1) &= 2 \\ -2x - 2 &= 2 \\ -2x &= 4 \\ x &= -2 \end{aligned}$$

This gives the point $(-2; 0)$.

Exercise 5 – 11: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x) = \frac{1}{x+4} - 2$
2. $g(x) = -\frac{5}{x} + 2$
3. $j(x) = \frac{2}{x-1} + 3$
4. $h(x) = \frac{3}{6-x} + 1$
5. $k(x) = \frac{5}{x+2} - \frac{1}{2}$

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1. 22RQ 2. 22RR 3. 22RS 4. 22RT 5. 22RV

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Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x+p} + q$. The asymptotes indicate the values of x for which the function does not exist. In other words, the values that are excluded from the domain and the range. The horizontal asymptote is the line $y = q$ and the vertical asymptote is the line $x = -p$.

Exercise 5 – 12: Asymptotes

Determine the asymptotes for each of the following functions:

1. $y = \frac{1}{x+4} - 2$
2. $y = -\frac{5}{x}$
3. $y = \frac{3}{2-x} + 1$
4. $y = \frac{1}{x} - 8$
5. $y = -\frac{2}{x-2}$

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1. 22RW 2. 22RX 3. 22RY 4. 22RZ 5. 22S2

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Axes of symmetry

There are two lines about which a hyperbola is symmetrical.

For the standard hyperbola $y = \frac{1}{x}$, we see that if we replace $x \Rightarrow y$ and $y \Rightarrow x$, we get $y = \frac{1}{x}$. Similarly, if we replace $x \Rightarrow -y$ and $y \Rightarrow -x$, the function remains the same. Therefore the function is symmetrical about the lines $y = x$ and $y = -x$.

For the shifted hyperbola $y = \frac{a}{x+p} + q$, the axes of symmetry intersect at the point $(-p; q)$.

To determine the axes of symmetry we define the two straight lines $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$. For the standard and shifted hyperbolic function, the gradient of one of the lines of symmetry is 1 and the gradient of the other line of symmetry is -1. The axes of symmetry are perpendicular to each other and the product of their gradients equals -1. Therefore we let $y_1 = x + c_1$ and $y_2 = -x + c_2$. We then substitute $(-p; q)$, the point of intersection of the axes of symmetry, into both equations to determine the values of c_1 and c_2.

Worked example 10: Axes of symmetry

QUESTION

Determine the axes of symmetry for $y = \frac{2}{x+1} - 2$.

SOLUTION

Step 1: Determine the point of intersection $(-p; q)$

From the equation we see that $p = 1$ and $q = -2$. So the axes of symmetry will intersect at $(-1; -2)$.

Step 2: Define two straight line equations

$$\begin{aligned} y_1 &= x + c_1 \\ y_2 &= -x + c_2 \end{aligned}$$

Step 3: Solve for c_1 and c_2

Use $(-1; -2)$ to solve for c_1:

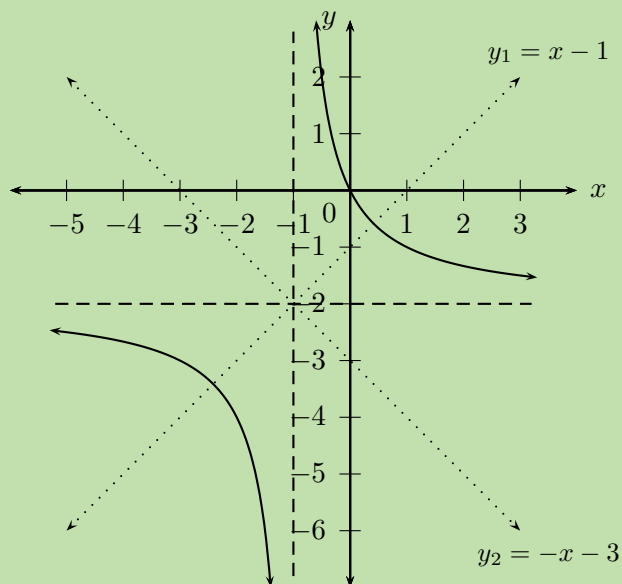
$$\begin{aligned} y_1 &= x + c_1 \\ -2 &= -1 + c_1 \\ -1 &= c_1 \end{aligned}$$

Use $(-1; -2)$ to solve for c_2:

$$\begin{aligned} y_2 &= -x + c_2 \\ -2 &= -(-1) + c_2 \\ -3 &= c_2 \end{aligned}$$

Step 4: Write the final answer

The axes of symmetry for $y = \frac{2}{x+1} - 2$ are the lines

$$\begin{aligned} y_1 &= x - 1 \\ y_2 &= -x - 3 \end{aligned}$$

Exercise 5 – 13: Axes of symmetry

1. Complete the following for $f(x)$ and $g(x)$:

   - Sketch the graph.
   - Determine $(-p; q)$.
   - Find the axes of symmetry.

   Compare $f(x)$ and $g(x)$ and also their axes of symmetry. What do you notice?

   a) $f(x) = \frac{2}{x}$
      $g(x) = \frac{2}{x} + 1$

   b) $f(x) = -\frac{3}{x}$
      $g(x) = -\frac{3}{x+1}$

   c) $f(x) = \frac{5}{x}$
      $g(x) = \frac{5}{x-1} - 1$

2. A hyperbola of the form $k(x) = \frac{a}{x+p} + q$ passes through the point $(4; 3)$. If the axes of symmetry intersect at $(-1; 2)$, determine the equation of $k(x)$.

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1a. 22S3 1b. 22S4 1c. 22S5 2. 22S6

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Sketching graphs of the form $f(x) = \frac{a}{x+p} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x+p} + q$, we need to calculate five characteristics:

- quadrants
- asymptotes
- y-intercept
- x-intercept
- domain and range

Worked example 11: Sketching a hyperbola

QUESTION

Sketch the graph of $y = \frac{2}{x+1} + 2$. Determine the intercepts, asymptotes and axes of symmetry. State the domain and range of the function.

SOLUTION

Step 1: Examine the equation of the form $y = \frac{a}{x+p} + q$

We notice that $a > 0$, therefore the graph will lie in the first and third quadrants.

Step 2: Determine the asymptotes

From the equation we know that $p = 1$ and $q = 2$.

Therefore the horizontal asymptote is the line $y = 2$ and the vertical asymptote is the line $x = -1$.

Step 3: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= \frac{2}{0+1} + 2 \\ &= 4 \end{aligned}$$

This gives the point $(0; 4)$.

Step 4: Determine the x-intercept

The x-intercept is obtained by letting $y = 0$:

$$\begin{aligned} 0 &= \frac{2}{x+1} + 2 \\ -2 &= \frac{2}{x+1} \end{aligned}$$

$$-2(x+1) = 2$$
$$-2x - 2 = 2$$
$$-2x = 4$$
$$x = -2$$

This gives the point $(-2;0)$.

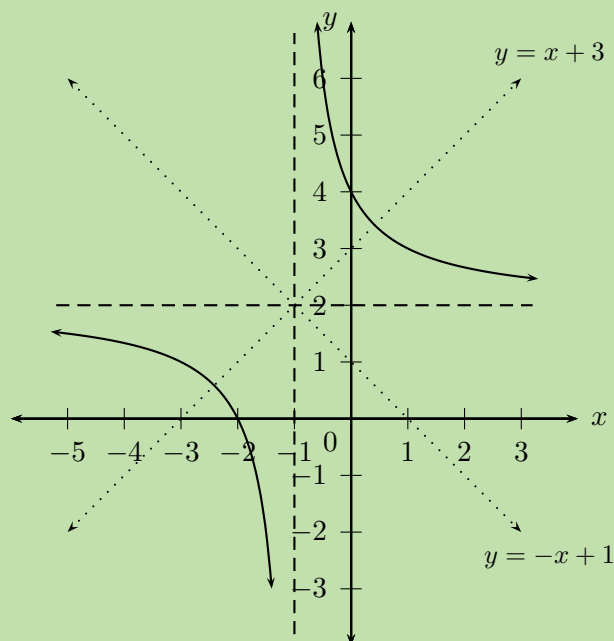
Step 5: Determine the axes of symmetry

Using $(-1;2)$ to solve for c_1:

$$y_1 = x + c_1$$
$$2 = -1 + c_1$$
$$3 = c_1$$

$$y_2 = -x + c_2$$
$$2 = -(-1) + c_2$$
$$1 = c_2$$

Therefore the axes of symmetry are $y = x + 3$ and $y = -x + 1$.

Step 6: Plot the points and sketch the graph

Step 7: State the domain and range

Domain: $\{x : x \in \mathbb{R}, x \neq -1\}$

Range: $\{y : y \in \mathbb{R}, y \neq 2\}$

Worked example 12: Sketching a hyperbola

QUESTION

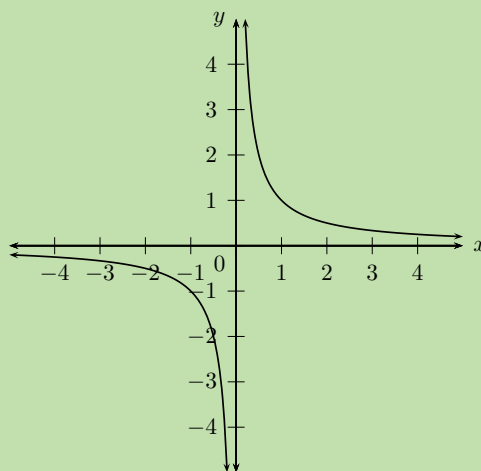
Use horizontal and vertical shifts to sketch the graph of $f(x) = \frac{1}{x-2} + 3$.

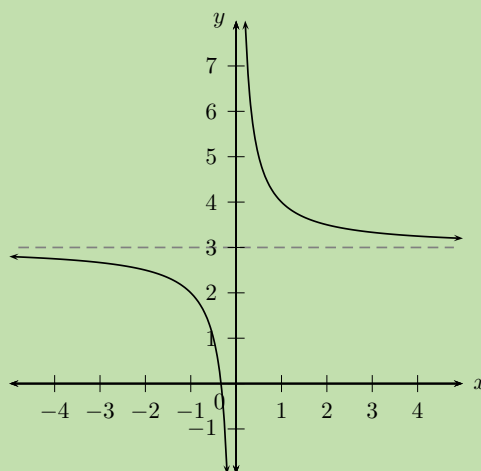
SOLUTION

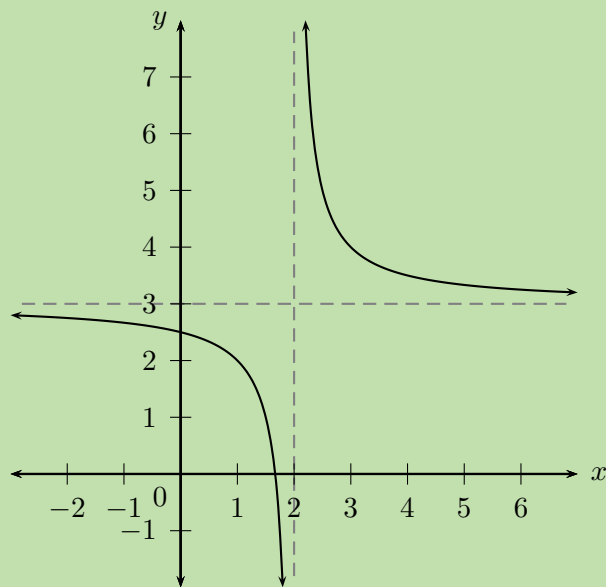
Step 1: Examine the equation of the form $y = \frac{a}{x+p} + q$

We notice that $a > 0$, therefore the graph will lie in the first and third quadrants.

Step 2: Sketch the standard hyperbola $y = \frac{1}{x}$

Start with a sketch of the standard hyperbola $g(x) = \frac{1}{x}$. The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

Step 3: Determine the vertical shift

From the equation we see that $q = 3$, which means $g(x)$ must shifted 3 units up. The horizontal asymptote is also shifted 3 units up to $y = 3$.

Step 4: Determine the horizontal shift

From the equation we see that $p = -2$, which means $g(x)$ must shifted 2 units to the right. The vertical asymptote is also shifted 2 units to the right.

Step 5: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} y &= \frac{1}{0-2} + 3 \\ &= 2\frac{1}{2} \end{aligned}$$

This gives the point $(0; 2\frac{1}{2})$.

Step 6: Determine the x-intercept

The x-intercept is obtained by letting $y = 0$:

$$\begin{aligned} 0 &= \frac{1}{x-2} + 3 \\ -3 &= \frac{1}{x-2} \\ -3(x-2) &= 1 \\ -3x + 6 &= 1 \\ -3x &= -5 \\ x &= \frac{5}{3} \end{aligned}$$

This gives the point $(\frac{5}{3}; 0)$.

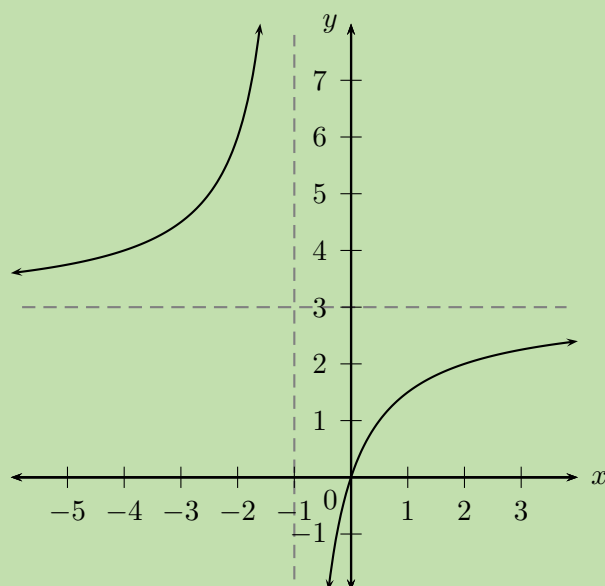
Step 7: Determine the domain and range

Domain: $\{x : x \in \mathbb{R}, x \neq 2\}$

Range: $\{y : y \in \mathbb{R}, y \neq 3\}$

Worked example 13: Finding the equation of a hyperbola from the graph

QUESTION

Use the graph below to determine the values of a, p and q for $y = \frac{a}{x+p} + q$.

SOLUTION

Step 1: Examine the graph and deduce the sign of a

We notice that the graph lies in the second and fourth quadrants, therefore $a < 0$.

Step 2: Determine the asymptotes

From the graph we see that the vertical asymptote is $x = -1$, therefore $p = 1$. The horizontal asymptote is $y = 3$, and therefore $q = 3$.

$$y = \frac{a}{x+1} + 3$$

Step 3: Determine the value of a

To determine the value of a we substitute a point on the graph, namely $(0; 0)$:

$$\begin{aligned} y &= \frac{a}{x+1} + 3 \\ 0 &= \frac{a}{0+1} + 3 \\ \therefore -3 &= a \end{aligned}$$

Step 4: Write the final answer

$$y = -\frac{3}{x+1} + 3$$

Exercise 5 – 14: Sketching graphs

1. Draw the graphs of the following functions and indicate:

   - asymptotes
   - intercepts, where applicable
   - axes of symmetry
   - domain and range

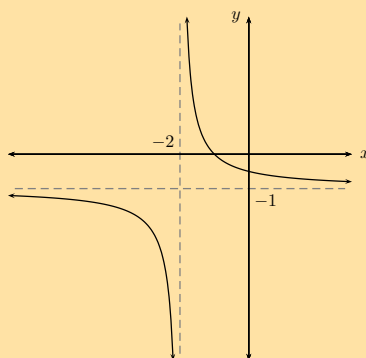
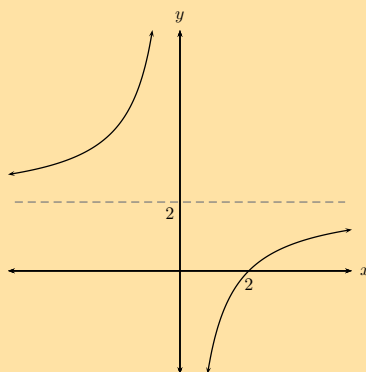
   a) $y = \frac{1}{x} + 2$

   b) $y = \frac{1}{x+4} - 2$

   c) $y = -\frac{1}{x+1} + 3$

   d) $y = -\frac{5}{x-2\frac{1}{2}} - 2$

   e) $y = \frac{8}{x-8} + 4$

2. Given the graph of the hyperbola of the form $y = \frac{1}{x+p} + q$, determine the values of p and q.

3. Given a sketch of the function of the form $y = \frac{a}{x+p} + q$, determine the values of a, p and q.

4. a) Draw the graph of $f(x) = -\frac{3}{x}$, $x > 0$.

   b) Determine the average gradient of the graph between $x = 1$ and $x = 3$.

   c) Is the gradient at $(\frac{1}{2}; -6)$ less than or greater than the average gradient between $x = 1$ and $x = 3$? Illustrate this on your graph.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22S7 1b. 22S8 1c. 22S9 1d. 22SB 1e. 22SC 2. 22SD
3. 22SF 4. 22SG

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5.4 Exponential functions EMBGS

Revision EMBGT

Functions of the form $y = ab^x + q$

Functions of the general form $y = ab^x + q$, for $b > 0$, are called exponential functions, where a, b and q are constants.

The effects of a, b and q on $f(x) = ab^x + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(x)$ is shifted vertically upwards by q units.
  - For $q < 0$, $f(x)$ is shifted vertically downwards by q units.
  - The horizontal asymptote is the line $y = q$.
- **The effect of a on shape**
  - For $a > 0$, $f(x)$ is increasing.
  - For $a < 0$, $f(x)$ is decreasing. The graph is reflected about the horizontal asymptote.
- **The effect of b on direction**

  Assuming $a > 0$:
  - If $b > 1$, $f(x)$ is an increasing function.
  - If $0 < b < 1$, $f(x)$ is a decreasing function.
  - If $b \leq 0$, $f(x)$ is not defined.

$b > 1$	$a < 0$	$a > 0$
$q > 0$		
$q < 0$		

$0 < b < 1$	$a < 0$	$a > 0$
$q > 0$		
$q < 0$		

Exercise 5 – 15: Revision

1. On separate axes, accurately draw each of the following functions:

   - Use tables of values if necessary.
   - Use graph paper if available.

   a) $y_1 = 3^x$

   b) $y_2 = -2 \times 3^x$

   c) $y_3 = 2 \times 3^x + 1$

   d) $y_4 = 3^x - 2$

2. Use your sketches of the functions given above to complete the following table (the first column has been completed as an example):

	y_1	y_2	y_3	y_4
value of q	$q = 0$			
effect of q	no vertical shift			
value of a	$a = 1$			
effect of a	increasing			
asymptote	x-axis, $y = 0$			
domain	$\{x : x \in \mathbb{R}\}$			
range	$\{y : y \in \mathbb{R}, y > 0\}$			

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22SH 1b. 22SJ 1c. 22SK 1d. 22SM 2. 22SN

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Functions of the form $y = ab^{(x+p)} + q$ EMBGV

We now consider exponential functions of the form $y = ab^{(x+p)} + q$ and the effects of parameter p.

See video: 22SP at www.everythingmaths.co.za

Investigation: The effects of a, p and q on an exponential graph

1. On the same system of axes, plot the following graphs:

   a) $y_1 = 2^x$
   b) $y_2 = 2^{(x-2)}$
   c) $y_3 = 2^{(x-1)}$
   d) $y_4 = 2^{(x+1)}$
   e) $y_5 = 2^{(x+2)}$

   Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
intercept(s)					
asymptote					
domain					
range					
effect of p					

2. On the same system of axes, plot the following graphs:

   a) $y_1 = 2^{(x-1)} + 2$

b) $y_2 = 3 \times 2^{(x-1)} + 2$

c) $y_3 = \frac{1}{2} \times 2^{(x-1)} + 2$

d) $y_4 = 0 \times 2^{(x-1)} + 2$

e) $y_5 = -3 \times 2^{(x-1)} + 2$

Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
intercept(s)					
asymptotes					
domain					
range					
effect of a					

The effect of the parameters on $y = ab^{x+p} + q$

The effect of p is a horizontal shift because all points are moved the same distance in the same direction (the entire graph slides to the left or to the right).

- For $p > 0$, the graph is shifted to the left by p units.
- For $p < 0$, the graph is shifted to the right by p units.

The effect of q is a vertical shift. The value of q also affects the horizontal asymptotes, the line $y = q$.

The value of a affects the shape of the graph and its position relative to the horizontal asymptote.

- For $a > 0$, the graph lies above the horizontal asymptote, $y = q$.
- For $a < 0$, the graph lies below the horizontal asymptote, $y = q$.

Discovering the characteristics

For functions of the general form: $f(x) = y = ab^{(x+p)} + q$:

Domain and range

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $f(x)$ is undefined.

The range of $f(x)$ depends on whether the value for a is positive or negative.

If $a > 0$ we have:

$$\begin{aligned} b^{(x+p)} &> 0 \\ ab^{(x+p)} &> 0 \\ ab^{(x+p)} + q &> q \\ f(x) &> q \end{aligned}$$

The range is therefore $\{y : y > q, y \in \mathbb{R}\}$.

Similarly, if $a < 0$, the range is $\{y : y < q, y \in \mathbb{R}\}$.

Worked example 14: Domain and range

QUESTION

State the domain and range for $g(x) = 5 \times 3^{(x+1)} - 1$.

SOLUTION

Step 1: Determine the domain

The domain is $\{x : x \in \mathbb{R}\}$ because there is no value of x for which $g(x)$ is undefined.

Step 2: Determine the range

The range of $g(x)$ can be calculated from:

$$\begin{aligned} 3^{(x+1)} &> 0 \\ 5 \times 3^{(x+1)} &> 0 \\ 5 \times 3^{(x+1)} - 1 &> -1 \\ \therefore g(x) &> -1 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) > -1\}$ or in interval notation $(-1; \infty)$.

Exercise 5 – 16: Domain and range

Give the domain and range for each of the following functions:

1. $y = \left(\frac{3}{2}\right)^{(x+3)}$
2. $f(x) = -5^{(x-2)} + 1$
3. $y + 3 = 2^{(x+1)}$
4. $y = n + 3^{(x-m)}$
5. $\frac{y}{2} = 3^{(x-1)} - 1$

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1. 22SQ 2. 22SR 3. 22SS 4. 22ST 5. 22SV

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Intercepts

The y-intercept:

To calculate the y-intercept we let $x = 0$. For example, the y-intercept of $g(x) = 3 \times 2^{(x+1)} + 2$ is determined by setting $x = 0$:

$$\begin{aligned} g(0) &= 3 \times 2^{(0+1)} + 2 \\ &= 3 \times 2 + 2 \\ &= 8 \end{aligned}$$

This gives the point $(0; 8)$.

The x-intercept:

To calculate the x-intercept we let $y = 0$. For example, the x-intercept of $g(x) = 3 \times 2^{(x+1)} + 2$ is determined by setting $y = 0$:

$$\begin{aligned} 0 &= 3 \times 2^{(x+1)} + 2 \\ -2 &= 3 \times 2^{(x+1)} \\ -\frac{2}{3} &= 2^{(x+1)} \end{aligned}$$

which has no real solutions. Therefore, the graph of $g(x)$ lies above the x-axis and does not have any x-intercepts.

Exercise 5 – 17: Intercepts

Determine the x- and y-intercepts for each of the following functions:

1. $f(x) = 2^{(x+1)} - 8$
2. $y = 2 \times 3^{(x-1)} - 18$
3. $y + 5^{(x+2)} = 5$
4. $y = \frac{1}{2}\left(\frac{3}{2}\right)^{(x+3)} - 0{,}75$

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1. 22SW 2. 22SX 3. 22SY 4. 22SZ

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Asymptote

Exponential functions of the form $y = ab^{(x+p)} + q$ have a horizontal asymptote, the line $y = q$.

Worked example 15: Asymptote

QUESTION

Determine the asymptote for $y = 5 \times 3^{(x+1)} - 1$.

SOLUTION

Step 1: Determine the asymptote

The asymptote of $g(x)$ can be calculated as:

$$\begin{aligned} 3^{(x+1)} &\neq 0 \\ 5 \times 3^{(x+1)} &\neq 0 \\ 5 \times 3^{(x+1)} - 1 &\neq -1 \\ \therefore y &\neq -1 \end{aligned}$$

Therefore the asymptote is the line $y = -1$.

Exercise 5 – 18: Asymptote

Give the asymptote for each of the following functions:

1. $y = -5^{(x+1)}$
2. $y = 3^{(x-2)} + 1$
3. $\left(\frac{3y}{2}\right) = 5^{(x+3)} - 1$
4. $y = 7^{(x+1)} - 2$
5. $\frac{y}{2} + 1 = 3^{(x+2)}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22T2 2. 22T3 3. 22T4 4. 22T5 5. 22T6

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Sketching graphs of the form $f(x) = ab^{(x+p)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x+p)} + q$, we need to determine five characteristics:

- shape
- y-intercept
- x-intercept
- asymptote
- domain and range

Worked example 16: Sketching an exponential graph

QUESTION

Sketch the graph of $2y = 10 \times 2^{(x+1)} - 5$.

Mark the intercept(s) and asymptote. State the domain and range of the function.

SOLUTION

Step 1: Examine the equation of the form $y = ab^{(x+p)} + q$

We notice that $a > 0$ and $b > 1$, therefore the function is increasing.

Step 2: Determine the y-intercept

The y-intercept is obtained by letting $x = 0$:

$$\begin{aligned} 2y &= 10 \times 2^{(0+1)} - 5 \\ &= 10 \times 2 - 5 \\ &= 15 \\ \therefore y &= 7\frac{1}{2} \end{aligned}$$

This gives the point $(0; 7\frac{1}{2})$.

Step 3: Determine the x-intercept

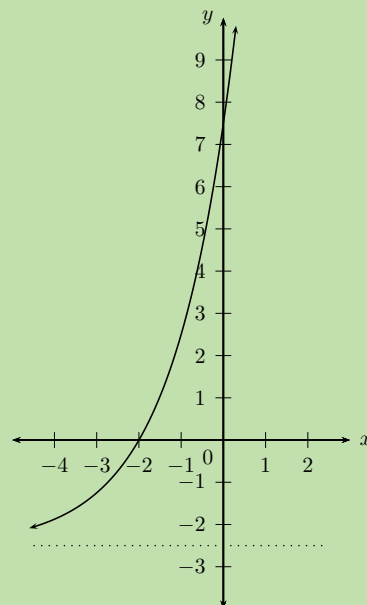
The x-intercept is obtained by letting $y = 0$:

$$\begin{aligned} 0 &= 10 \times 2^{(x+1)} - 5 \\ 5 &= 10 \times 2^{(x+1)} \\ \frac{1}{2} &= 2^{(x+1)} \\ 2^{-1} &= 2^{(x+1)} \\ \therefore -1 &= x + 1 \quad \text{(same base)} \\ -2 &= x \end{aligned}$$

This gives the point $(-2; 0)$.

Step 4: Determine the asymptote

The horizontal asymptote is the line $y = -\frac{5}{2}$.

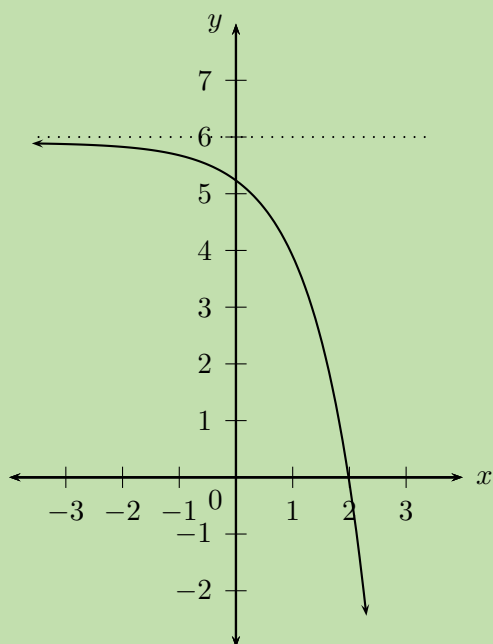
Step 5: Plot the points and sketch the graph

Step 6: State the domain and range

Domain: $\{x : x \in \mathbb{R}\}$

Range: $\{y : y > -\frac{5}{2}, y \in \mathbb{R}\}$

Worked example 17: Finding the equation of an exponential function from a graph

QUESTION

Use the given graph of $y = -2 \times 3^{(x+p)} + q$ to determine the values of p and q.

SOLUTION

Step 1: Examine the equation of the form $y = ab^{(x+p)} + q$

From the graph we see that the function is decreasing. We also note that $a = -2$ and $b = 3$. We need to solve for p and q.

Step 2: Use the asymptote to determine q

The horizontal asymptote $y = 6$ is given, therefore we know that $q = 6$.

$$y = -2 \times 3^{(x+p)} + 6$$

Step 3: Use the x-intercept to determine p

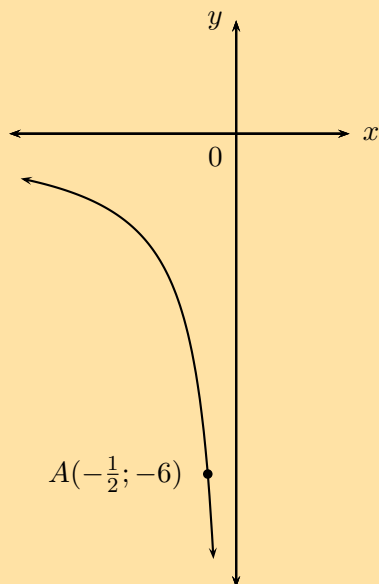
Substitute $(2; 0)$ into the equation and solve for p:

$$\begin{aligned} y &= -2 \times 3^{(x+p)} + 6 \\ 0 &= -2 \times 3^{(2+p)} + 6 \\ -6 &= -2 \times 3^{(2+p)} \\ 3 &= 3^{(2+p)} \\ \therefore 1 &= 2 + p \quad \text{(same base)} \\ \therefore p &= -1 \end{aligned}$$

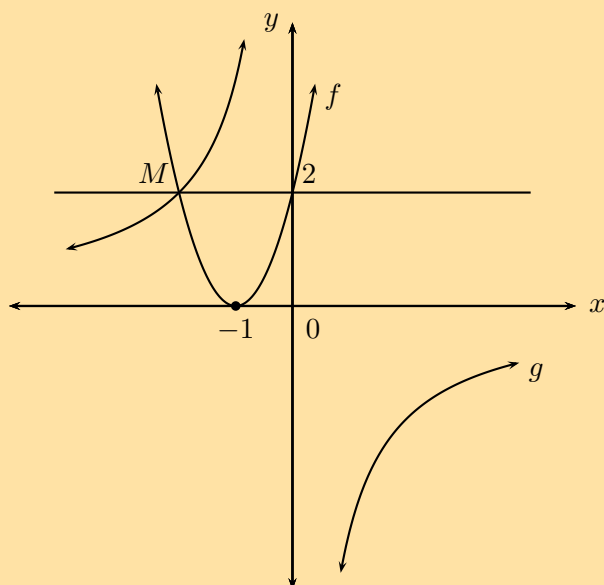
Step 4: Write the final answer

$$y = -2 \times 3^{(x-1)} + 6$$

Exercise 5 – 19: Mixed exercises

1. Given the graph of the hyperbola of the form $h(x) = \frac{k}{x}$, $x < 0$, which passes though the point $A(-\frac{1}{2}; -6)$.

   a) Show that $k = 3$.
   b) Write down the equation for the new function formed if $h(x)$:
      i. is shifted 3 units vertically upwards
      ii. is shifted to the right by 3 units
      iii. is reflected about the y-axis
      iv. is shifted so that the asymptotes are $x = 0$ and $y = -\frac{1}{4}$
      v. is shifted upwards to pass through the point $(-1; 1)$
      vi. is shifted to the left by 2 units and 1 unit vertically downwards (for $x < 0$)

2. Given the graphs of $f(x) = a(x + p)^2$ and $g(x) = \frac{a}{x}$.

   The axis of symmetry for $f(x)$ is $x = -1$ and $f(x)$ and $g(x)$ intersect at point M. The line $y = 2$ also passes through M.

Determine:

a) the coordinates of M

b) the equation of $g(x)$

c) the equation of $f(x)$

d) the values for which $f(x) < g(x)$

e) the range of $f(x)$

3. On the same system of axes, sketch:

   a) the graphs of $k(x) = 2(x+\frac{1}{2})^2 - 4\frac{1}{2}$ and $h(x) = 2^{(x+\frac{1}{2})}$. Determine all intercepts, turning point(s) and asymptotes.

   b) the reflection of $h(x)$ about the x-axis. Label this function as $j(x)$.

4. Sketch the graphs of $y = ax^2 + bx + c$ for:

   a) $a < 0$, $b > 0$, $b^2 < 4ac$

   b) $a > 0$, $b > 0$, one root $= 0$

5. On separate systems of axes, sketch the graphs:

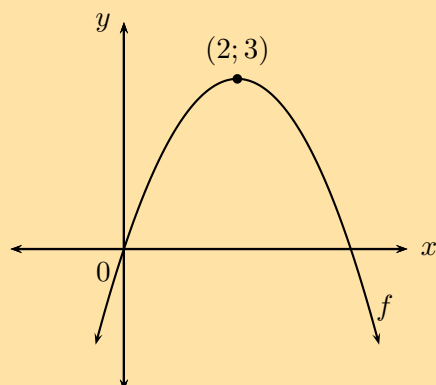
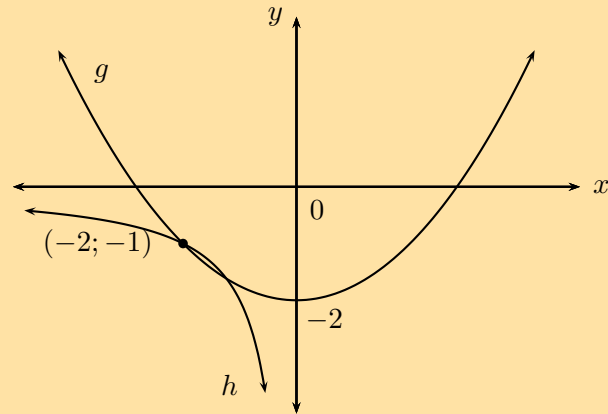
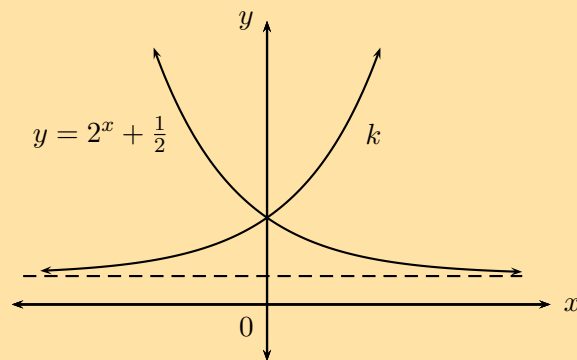
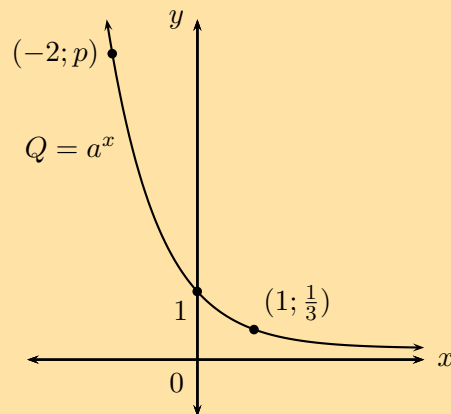
   $y = \frac{2}{x-2}$

   $y = \frac{2}{x} - 2$

   $y = -2^{(x-2)}$

6. For the diagrams shown below, determine:

   - the equations of the functions; $f(x) = a(x+p)^2 + q$, $g(x) = ax^2 + q$, $h(x) = \frac{a}{x}, x < 0$ and $k(x) = b^x + q$
   - the axes of symmetry of each function
   - the domain and range of each function

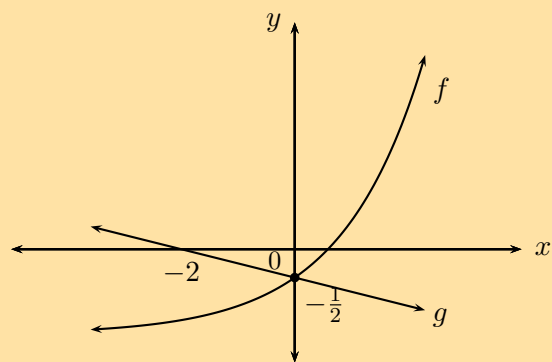
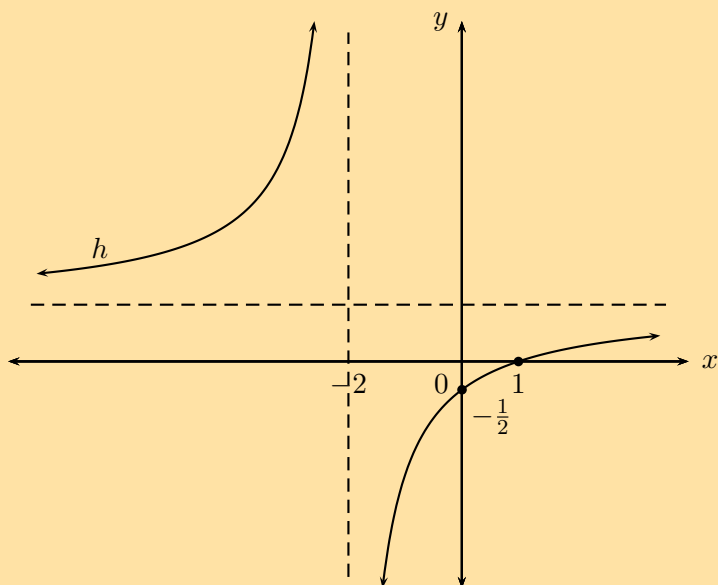
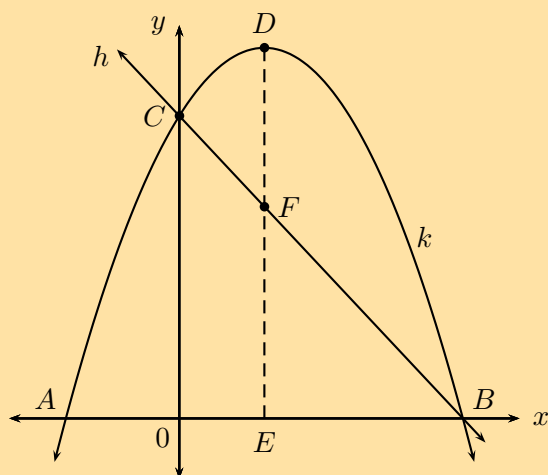
   a)

b)

c)

7. Given the graph of the function $Q(x) = a^x$.

a) Show that $a = \frac{1}{3}$.

b) Find the value of p if the point $(-2;p)$ is on Q.

c) Calculate the average gradient of the curve between $x = -2$ and $x = 1$.

d) Determine the equation of the new function formed if Q is shifted 2 units vertically downwards and 2 units to the left.

8. Find the equation for each of the functions shown below:

a) $f(x) = 2^x + q$
$g(x) = mx + c$

b) $h(x) = \frac{k}{x+p} + q$

9. Given: the graph of $k(x) = -x^2 + 3x + 10$ with turning point at D. The graph of the straight line $h(x) = mx + c$ passing through points B and C is also shown.

Determine:

a) the lengths AO, OB, OC and DE
b) the equation of DE
c) the equation of $h(x)$
d) the x-values for which $k(x) < 0$

e) the x-values for which $k(x) \geq h(x)$

f) the length of DF

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22T7 2. 22T8 3. 22T9 4. 22TB 5. 22TC 6a. 22TD
6b. 22TF 6c. 22TG 7. 22TH 8a. 22TJ 8b. 22TK 9. 22TM

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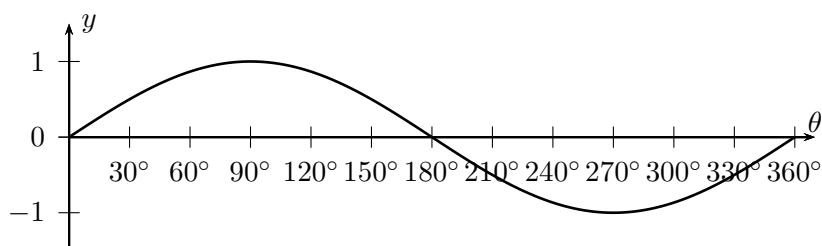
IMPORTANT: Trigonometric functions are examined in PAPER 2.

5.5 The sine function EMBGW

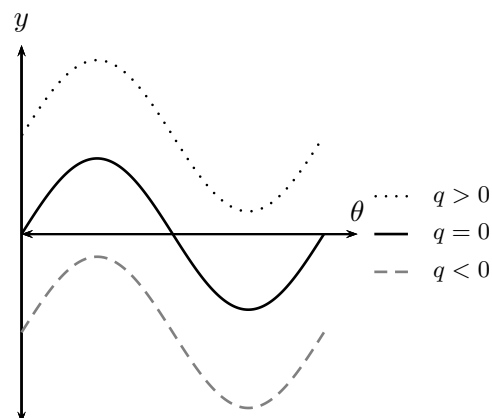
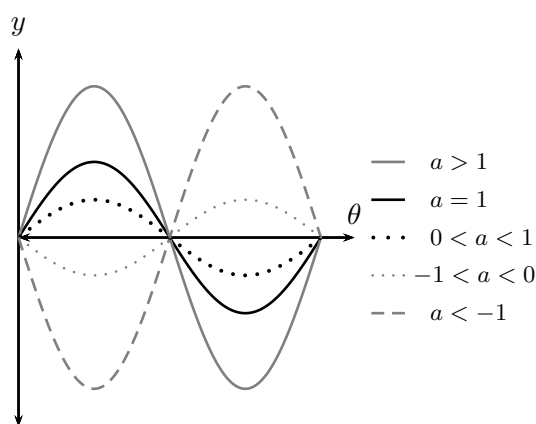
Revision EMBGX

Functions of the form $y = \sin\theta$ for $0° \leq \theta \leq 360°$

- Period of one complete wave is $360°$.
- Amplitude is the maximum height of the wave above and below the x-axis and is always positive. Amplitude $= 1$.
- Domain: $[0°; 360°]$

  For $y = \sin\theta$, the domain is $\{\theta : \theta \in \mathbb{R}\}$, however in this case, the domain has been restricted to the interval $0° \leq \theta \leq 360°$.
- Range: $[-1; 1]$
- x-intercepts: $(0°; 0)$, $(180°; 0)$, $(360°; 0)$
- y-intercept: $(0°; 0)$
- Maximum turning point: $(90°; 1)$
- Minimum turning point: $(270°; -1)$

Functions of the form $y = a\sin\theta + q$

The effects of a and q on $f(\theta) = a\sin\theta + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(\theta)$ is shifted vertically upwards by q units.
  - For $q < 0$, $f(\theta)$ is shifted vertically downwards by q units.

- **The effect of a on shape**
  - For $a > 1$, the amplitude of $f(\theta)$ increases.
  - For $0 < a < 1$, the amplitude of $f(\theta)$ decreases.
  - For $a < 0$, there is a reflection about the x-axis.
  - For $-1 < a < 0$, there is a reflection about the x-axis and the amplitude decreases.
  - For $a < -1$, there is a reflection about the x-axis and the amplitude increases.

Exercise 5 – 20: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$.

- Use tables of values if necessary.
- Use graph paper if available.

For each function also determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $y_1 = \sin\theta$
2. $y_2 = -2\sin\theta$
3. $y_3 = \sin\theta + 1$
4. $y_4 = \frac{1}{2}\sin\theta - 1$

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1. 22TN 2. 22TP 3. 22TQ 4. 22TR

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Functions of the form $y = \sin k\theta$ EMBGY

We now consider cosine functions of the form $y = \sin k\theta$ and the effects of k.

Investigation: The effects of k on a sine graph

1. Complete the following table for $y_1 = \sin\theta$ for $-360° \leq \theta \leq 360°$:

θ	$-360°$	$-270°$	$-180°$	$-90°$	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$									

2. Use the table of values to plot the graph of $y_1 = \sin\theta$ for $-360° \leq \theta \leq 360°$.
3. On the same system of axes, plot the following graphs:
   a) $y_2 = \sin(-\theta)$
   b) $y_3 = \sin 2\theta$
   c) $y_4 = \sin\frac{\theta}{2}$
4. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4
period				
amplitude				
domain				
range				
maximum turning points				
minimum turning points				
y-intercept(s)				
x-intercept(s)				
effect of k				

5. What do you notice about $y_1 = \sin\theta$ and $y_2 = \sin(-\theta)$?
6. Is $\sin(-\theta) = -\sin\theta$ a true statement? Explain your answer.
7. Can you deduce a formula for determining the period of $y = \sin k\theta$?

The effect of the parameter on $y = \sin k\theta$

The value of k affects the period of the sine function. If k is negative, then the graph is reflected about the y-axis.

- **For $k > 0$**:

  For $k > 1$, the period of the sine function decreases.

  For $0 < k < 1$, the period of the sine function increases.

- **For $k < 0$**:

  For $-1 < k < 0$, the graph is reflected about the y-axis and the period increases.

  For $k < -1$, the graph is reflected about the y-axis and the period decreases.

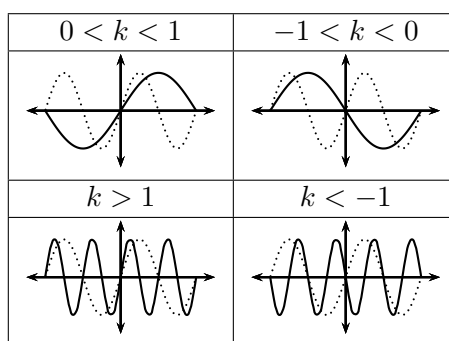
Negative angles:

$$\sin(-\theta) = -\sin\theta$$

Calculating the period:

To determine the period of $y = \sin k\theta$ we use,

$$\text{Period} = \frac{360°}{|k|}$$

where $|k|$ is the absolute value of k (this means that k is always considered to be positive).

Worked example 18: Sine function

QUESTION

1. Sketch the following functions on the same set of axes for $-180° \leq \theta \leq 180°$.
   a) $y_1 = \sin\theta$
   b) $y_2 = \sin\frac{3\theta}{2}$
2. For each function determine the following:
   a) Period
   b) Amplitude
   c) Domain and range
   d) x- and y-intercepts
   e) Maximum and minimum turning points

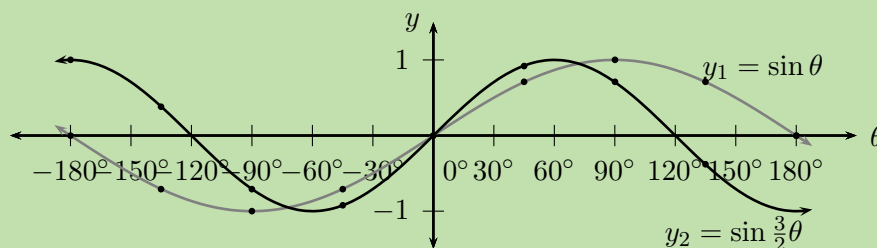
SOLUTION

Step 1: Examine the equations of the form $y = \sin k\theta$

Notice that $k > 1$ for $y_2 = \sin\frac{3\theta}{2}$, therefore the period of the graph decreases.

Step 2: Complete a table of values

θ	$-180°$	$-135°$	$-90°$	$-45°$	$0°$	$45°$	$90°$	$135°$	$180°$
$\sin\theta$	0	$-0{,}71$	-1	$-0{,}71$	0	0,71	1	0,71	0
$\sin\frac{3\theta}{2}$	1	0,38	$-0{,}71$	$-0{,}92$	0	0,92	0,71	$-0{,}38$	-1

Step 3: Sketch the sine graphs

Step 4: Complete the table

	$y_1 = \sin\theta$	$y_2 = \sin\frac{3\theta}{2}$
period	$360°$	$240°$
amplitude	1	1
domain	$[-180°; 180°]$	$[-180°; 180°]$
range	$[-1; 1]$	$[-1; 1]$
maximum turning points	$(90°; 1)$	$(-180°; 1)$ and $(60°; 1)$
minimum turning points	$(-90°; -1)$	$(-60°; -1)$ and $(180°; 1)$
y-intercept(s)	$(0°; 0)$	$(0°; 0)$
x-intercept(s)	$(-180°; 0)$, $(0°; 0)$ and $(180°; 0)$	$(-120°; 0)$, $(0°; 0)$ and $(120°; 0)$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \sin k\theta$:

Domain and range

The domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value for θ for which $f(\theta)$ is undefined.

The range is $\{f(\theta) : -1 \leq f(\theta) \leq 1, f(\theta) \in \mathbb{R}\}$ or $[-1; 1]$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0°$ and solving for $f(\theta)$.

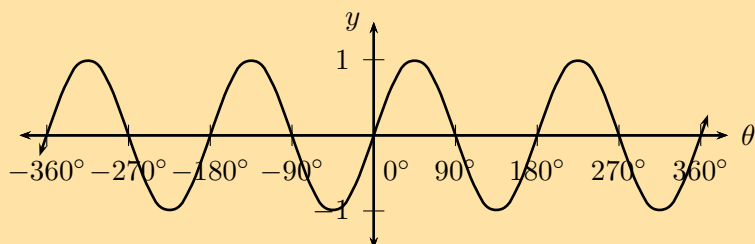
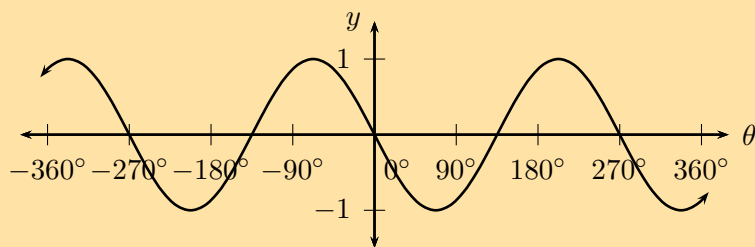
$$\begin{aligned} y &= \sin k\theta \\ &= \sin 0° \\ &= 0 \end{aligned}$$

This gives the point $(0°; 0)$.

Exercise 5 – 21: Sine functions of the form $y = \sin k\theta$

1. Sketch the following functions for $-180° \leq \theta \leq 180°$ and for each graph determine:
   - Period
   - Amplitude
   - Domain and range
   - x- and y-intercepts
   - Maximum and minimum turning points

   a) $f(\theta) = \sin 3\theta$
   b) $g(\theta) = \sin \frac{\theta}{3}$
   c) $h(\theta) = \sin(-2\theta)$
   d) $k(\theta) = \sin \frac{3\theta}{4}$

2. For each graph of the form $f(\theta) = \sin k\theta$, determine the value of k:

   a)

b)

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1a. 22TS 1b. 22TT 1c. 22TV 1d. 22TW 2a. 22TX 2b. 22TY

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Functions of the form $y = \sin(\theta + p)$ EMBGZ

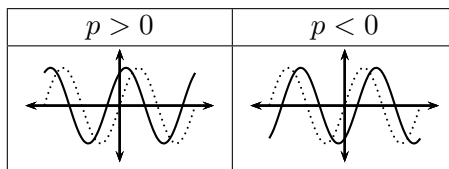
Investigation: The effects of p on a sine graph

1. On the same system of axes, plot the following graphs for $-360° \leq \theta \leq 360°$:
   a) $y_1 = \sin\theta$
   b) $y_2 = \sin(\theta - 90°)$
   c) $y_3 = \sin(\theta - 60°)$
   d) $y_4 = \sin(\theta + 90°)$
   e) $y_5 = \sin(\theta + 180°)$
2. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
period					
amplitude					
domain					
range					
maximum turning points					
minimum turning points					
y-intercept(s)					
x-intercept(s)					
effect of p					

The effect of the parameter on $y = \sin(\theta + p)$

The effect of p on the sine function is a horizontal shift, also called a phase shift; the entire graph slides to the left or to the right.

- For $p > 0$, the graph of the sine function shifts to the left by p.
- For $p < 0$, the graph of the sine function shifts to the right by p.

Worked example 19: Sine function

QUESTION

1. Sketch the following functions on the same set of axes for $-360° \leq \theta \leq 360°$.
   a) $y_1 = \sin\theta$
   b) $y_2 = \sin(\theta - 30°)$
2. For each function determine the following:
   a) Period
   b) Amplitude
   c) Domain and range
   d) x- and y-intercepts
   e) Maximum and minimum turning points

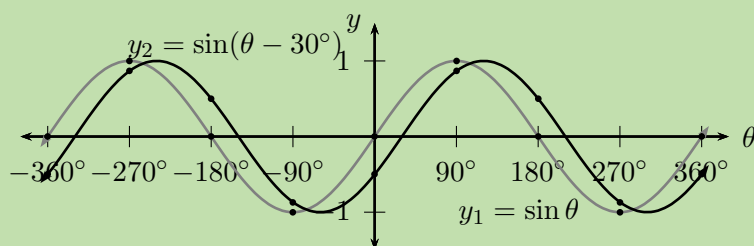
SOLUTION

Step 1: Examine the equations of the form $y = \sin(\theta + p)$

Notice that for $y_1 = \sin\theta$ we have $p = 0$ (no phase shift) and for $y_2 = \sin(\theta - 30°)$, $p < 0$ therefore the graph shifts to the right by $30°$.

Step 2: Complete a table of values

θ	$-360°$	$-270°$	$-180°$	$-90°$	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$	0	1	0	-1	0	1	0	-1	0
$\sin(\theta - 30°)$	$-0{,}5$	$0{,}87$	$0{,}5$	$-0{,}87$	$-0{,}5$	$0{,}87$	$0{,}5$	$-0{,}87$	$-0{,}5$

Step 3: Sketch the sine graphs

Step 4: Complete the table

	$y_1 = \sin\theta$	$y_2 = \sin(\theta - 30°)$
period	$360°$	$360°$
amplitude	1	1
domain	$[-360°; 360°]$	$[-360°; 360°]$
range	$[-1; 1]$	$[-1; 1]$
maximum turning points	$(-270°; 1)$ and $(90°; 1)$	$(-240°; 1)$ and $(120°; 1)$
minimum turning points	$(-90°; -1)$ and $(270°; -1)$	$(-60°; -1)$ and $(300°; -1)$
y-intercept(s)	$(0°; 0)$	$(0°; -\frac{1}{2})$
x-intercept(s)	$(-360°; 0)$, $(-180°; 0)$, $(0°; 0)$, $(180°; 0)$ and $(360°; 0)$	$(-330°; 0)$, $(-150°; 0)$, $(30°; 0)$ and $(210°; 0)$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \sin(\theta + p)$:

Domain and range

The domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value for θ for which $f(\theta)$ is undefined.

The range is $\{f(\theta) : -1 \leq f(\theta) \leq 1, f(\theta) \in \mathbb{R}\}$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0°$ and solving for $f(\theta)$.

Exercise 5 – 22: Sine functions of the form $y = \sin(\theta + p)$

Sketch the following functions for $-360° \leq \theta \leq 360°$.

For each function, determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $f(\theta) = \sin(\theta + 30°)$
2. $g(\theta) = \sin(\theta - 45°)$
3. $h(\theta) = \sin(\theta + 60°)$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22TZ 2. 22V2 3. 22V3

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Sketching sine graphs EMBH2

Worked example 20: Sketching a sine graph

QUESTION

Sketch the graph of $f(\theta) = \sin(45° - \theta)$ for $0° \leq \theta \leq 360°$.

SOLUTION

Step 1: Examine the form of the equation

Write the equation in the form $y = \sin(\theta + p)$.

$$\begin{aligned} f(\theta) &= \sin(45° - \theta) \\ &= \sin(-\theta + 45°) \\ &= \sin\left(-(\theta - 45°)\right) \\ &= -\sin(\theta - 45°) \end{aligned}$$

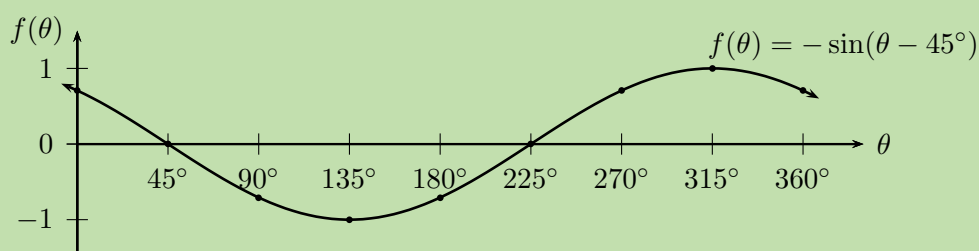
To draw a graph of the above function, we know that the standard sine graph, $y = \sin\theta$,

must:

- be reflected about the x-axis
- be shifted to the right by $45°$

Step 2: Complete a table of values

θ	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$f(\theta)$	$0{,}71$	0	$-0{,}71$	-1	$-0{,}71$	0	$0{,}71$	1	$0{,}71$

Step 3: Plot the points and join with a smooth curve

Period: $360°$
Amplitude: 1
Domain: $[-360°; 360°]$
Range: $[-1; 1]$
Maximum turning point: $(315°; 1)$
Minimum turning point: $(135°; -1)$
y-intercepts: $(0°; 0{,}71)$
x-intercept: $(45°; 0)$ and $(225°; 0)$

Worked example 21: Sketching a sine graph

QUESTION

Sketch the graph of $f(\theta) = \sin(3\theta + 60°)$ for $0° \leq \theta \leq 180°$.

SOLUTION

Step 1: Examine the form of the equation

Write the equation in the form $y = \sin k(\theta + p)$.

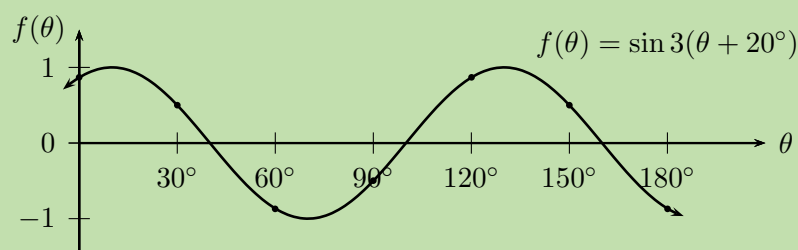
$$\begin{aligned} f(\theta) &= \sin(3\theta + 60°) \\ &= \sin 3(\theta + 20°) \end{aligned}$$

To draw a graph of the above equation, the standard sine graph, $y = \sin\theta$, must be changed in the following ways:

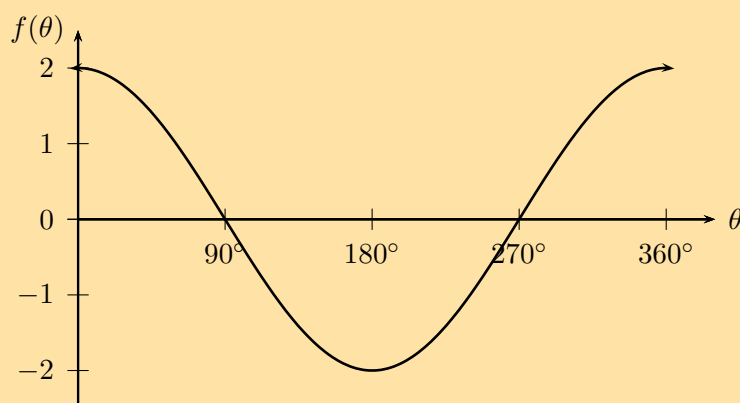
- decrease the period by a factor of 3;
- shift to the left by $20°$.

Step 2: Complete a table of values

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$
$f(\theta)$	0,87	0,5	−0,87	−0,5	0,87	0,5	−0,87

Step 3: Plot the points and join with a smooth curve

Period: $120°$
Amplitude: 1
Domain: $[0°; 180°]$
Range: $[-1; 1]$
Maximum turning point: $(10°; 1)$ and $(130°; 1)$
Minimum turning point: $(70°; -1)$
y-intercept: $(0°; 0{,}87)$
x-intercepts: $(40°; 0)$, $(100°; 0)$ and $(160°; 0)$

Exercise 5 – 23: The sine function

1. Sketch the following graphs on separate axes:
   a) $y = 2\sin\frac{\theta}{2}$ for $-360° \leq \theta \leq 360°$
   b) $f(\theta) = \frac{1}{2}\sin(\theta - 45°)$ for $-90° \leq \theta \leq 90°$
   c) $y = \sin(\theta + 90°) + 1$ for $0° \leq \theta \leq 360°$
   d) $y = \sin(-\frac{3\theta}{2})$ for $-180° \leq \theta \leq 180°$
   e) $y = \sin(30° - \theta)$ for $-360° \leq \theta \leq 360°$
2. Given the graph of the function $y = a\sin(\theta + p)$, determine the values of a and p.

Can you describe this graph in terms of $\cos\theta$?

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1a. 22V4 1b. 22V5 1c. 22V6 1d. 22V7 1e. 22V8 2. 22V9

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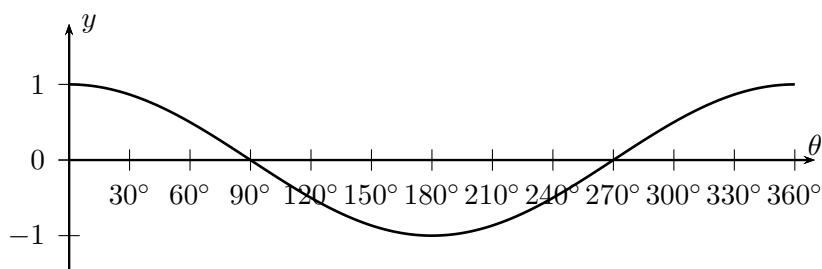
5.6 The cosine function EMBH3

Revision EMBH4

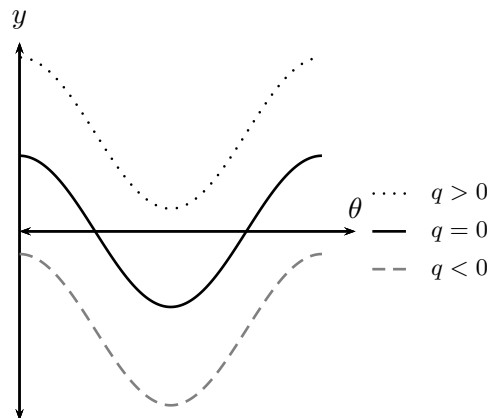
Functions of the form $y = \cos\theta$ for $0° \leq \theta \leq 360°$

- The period is $360°$ and the amplitude is 1.
- Domain: $[0°; 360°]$

  For $y = \cos\theta$, the domain is $\{\theta : \theta \in \mathbb{R}\}$, however in this case, the domain has been restricted to the interval $0° \leq \theta \leq 360°$.
- Range: $[-1; 1]$
- x-intercepts: $(90°; 0)$, $(270°; 0)$
- y-intercept: $(0°; 1)$
- Maximum turning points: $(0°; 1)$, $(360°; 1)$
- Minimum turning point: $(180°; -1)$

Functions of the form $y = a\cos\theta + q$

Cosine functions of the general form $y = a\cos\theta + q$, where a and q are constants.

The effects of a and q on $f(\theta) = a\cos\theta + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(\theta)$ is shifted vertically upwards by q units.
  - For $q < 0$, $f(\theta)$ is shifted vertically downwards by q units.

- **The effect of a on shape**
  - For $a > 1$, the amplitude of $f(\theta)$ increases.
  - For $0 < a < 1$, the amplitude of $f(\theta)$ decreases.
  - For $a < 0$, there is a reflection about the x-axis.
  - For $-1 < a < 0$, there is a reflection about the x-axis and the amplitude decreases.
  - For $a < -1$, there is a reflection about the x-axis and the amplitude increases.

Exercise 5 – 24: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$:

- Use tables of values if necessary.
- Use graph paper if available.

For each function in the previous problem determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $y_1 = \cos\theta$
2. $y_2 = -3\cos\theta$
3. $y_3 = \cos\theta + 2$
4. $y_4 = \frac{1}{2}\cos\theta - 1$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22VB 2. 22VC 3. 22VD 4. 22VF

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Functions of the form $y = \cos(k\theta)$ EMBH5

We now consider cosine functions of the form $y = \cos k\theta$ and the effects of k.

Investigation: The effects of k on a cosine graph

1. Complete the following table for $y_1 = \cos\theta$ for $-360° \leq \theta \leq 360°$:

θ	$-360°$	$-300°$	$-240°$	$-180°$	$-120°$	$-60°$	$0°$
$\cos\theta$							
θ	$60°$	$120°$	$180°$	$240°$	$300°$	$360°$	
$\cos\theta$							

2. Use the table of values to plot the graph of $y_1 = \cos\theta$ for $-360° \leq \theta \leq 360°$.
3. On the same system of axes, plot the following graphs:
   a) $y_2 = \cos(-\theta)$
   b) $y_3 = \cos 3\theta$
   c) $y_4 = \cos\frac{3\theta}{4}$
4. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4
period				
amplitude				
domain				
range				
maximum turning points				
minimum turning points				
y-intercept(s)				
x-intercept(s)				
effect of k				

5. What do you notice about $y_1 = \cos\theta$ and $y_2 = \cos(-\theta)$?
6. Is $\cos(-\theta) = -\cos\theta$ a true statement? Explain your answer.
7. Can you deduce a formula for determining the period of $y = \cos k\theta$?

The effect of the parameter k on $y = \cos k\theta$

The value of k affects the period of the cosine function.

- **For $k > 0$:**
  For $k > 1$, the period of the cosine function decreases.
  For $0 < k < 1$, the period of the cosine function increases.
- **For $k < 0$:**
  For $-1 < k < 0$, the period increases.
  For $k < -1$, the period decreases.

Negative angles:

$$\cos(-\theta) = \cos\theta$$

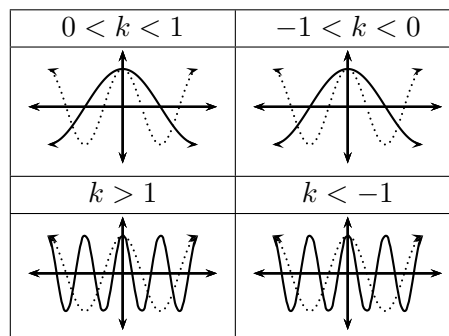
Notice that for negative values of θ, the graph is **not** reflected about the x-axis.

Calculating the period:

To determine the period of $y = \cos k\theta$ we use,

$$\text{Period} = \frac{360°}{|k|}$$

where $|k|$ is the absolute value of k.

Worked example 22: Cosine function

QUESTION

1. Sketch the following functions on the same set of axes for $-180° \leq \theta \leq 180°$.

   a) $y_1 = \cos\theta$
   b) $y_2 = \cos\frac{\theta}{2}$

2. For each function determine the following:

   a) Period
   b) Amplitude
   c) Domain and range
   d) x- and y-intercepts
   e) Maximum and minimum turning points

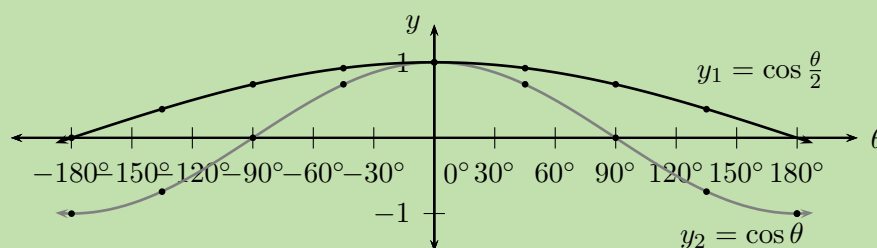
SOLUTION

Step 1: Examine the equations of the form $y = \cos k\theta$

Notice that for $y_2 = \cos\frac{\theta}{2}$, $k < 1$ therefore the period of the graph increases.

Step 2: Complete a table of values

θ	$-180°$	$-135°$	$-90°$	$-45°$	$0°$	$45°$	$90°$	$135°$	$180°$
$\cos\theta$	-1	$-0{,}71$	0	$0{,}71$	1	$0{,}71$	0	$-0{,}71$	-1
$\cos\frac{\theta}{2}$	0	$0{,}38$	$0{,}71$	$0{,}92$	1	$0{,}92$	$0{,}71$	$0{,}38$	0

Step 3: Sketch the cosine graphs

Step 4: Complete the table

	$y_1 = \cos\theta$	$y_2 = \cos\frac{\theta}{2}$
period	$360°$	$720°$
amplitude	1	1
domain	$[-180°; 180°]$	$[-180°; 180°]$
range	$[-1; 1]$	$[0; 1]$
maximum turning points	$(0°; 1)$	$(0°; 1)$
minimum turning points	$(-180°; -1)$ and $(180°; -1)$	none
y-intercept(s)	$(0°; 1)$	$(0°; 1)$
x-intercept(s)	$(-90°; 0)$ and $(90°; 0)$	$(-180°; 0)$ and $(180°; 0)$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \cos k\theta$:

Domain and range

The domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value for θ for which $f(\theta)$ is undefined.

The range is $\{f(\theta) : -1 \leq f(\theta) \leq 1, f(\theta) \in \mathbb{R}\}$ or $[-1; 1]$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0°$ and solving for $f(\theta)$.

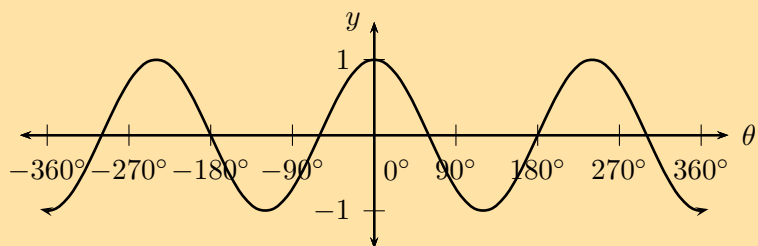
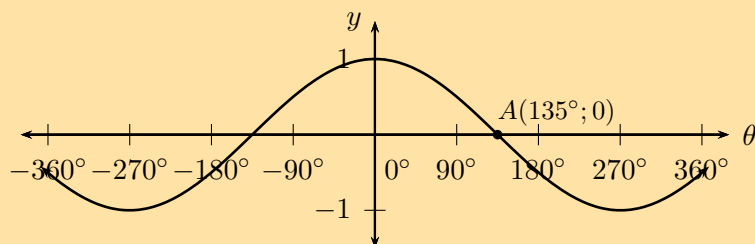
$$\begin{aligned} y &= \cos k\theta \\ &= \cos 0° \\ &= 1 \end{aligned}$$

This gives the point $(0°; 1)$.

Exercise 5 – 25: Cosine functions of the form $y = \cos k\theta$

1. Sketch the following functions for $-180° \leq \theta \leq 180°$. For each graph determine:
   - Period
   - Amplitude
   - Domain and range
   - x- and y-intercepts
   - Maximum and minimum turning points

   a) $f(\theta) = \cos 2\theta$
   b) $g(\theta) = \cos \frac{\theta}{3}$
   c) $h(\theta) = \cos(-2\theta)$
   d) $k(\theta) = \cos \frac{3\theta}{4}$

2. For each graph of the form $f(\theta) = \cos k\theta$, determine the value of k:

   a)

b)

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1a. 22VG 1b. 22VH 1c. 22VJ 1d. 22VK 2a. 22VM 2b. 22VN

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Functions of the form $y = \cos(\theta + p)$ EMBH6

We now consider cosine functions of the form $y = \cos(\theta + p)$ and the effects of parameter p.

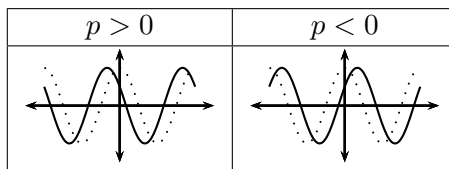
Investigation: The effects of p on a cosine graph

1. On the same system of axes, plot the following graphs for $-360° \leq \theta \leq 360°$:
   a) $y_1 = \cos\theta$
   b) $y_2 = \cos(\theta - 90°)$
   c) $y_3 = \cos(\theta - 60°)$
   d) $y_4 = \cos(\theta + 90°)$
   e) $y_5 = \cos(\theta + 180°)$
2. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
period					
amplitude					
domain					
range					
maximum turning points					
minimum turning points					
y-intercept(s)					
x-intercept(s)					
effect of p					

The effect of the parameter on $y = \cos(\theta + p)$

The effect of p on the cosine function is a horizontal shift (or phase shift); the entire graph slides to the left or to the right.

- For $p > 0$, the graph of the cosine function shifts to the left by p degrees.
- For $p < 0$, the graph of the cosine function shifts to the right by p degrees.

Worked example 23: Cosine function

QUESTION

1. Sketch the following functions on the same set of axes for $-360° \leq \theta \leq 360°$.
   a) $y_1 = \cos\theta$
   b) $y_2 = \cos(\theta + 30°)$
2. For each function determine the following:
   a) Period
   b) Amplitude
   c) Domain and range
   d) x- and y-intercepts
   e) Maximum and minimum turning points

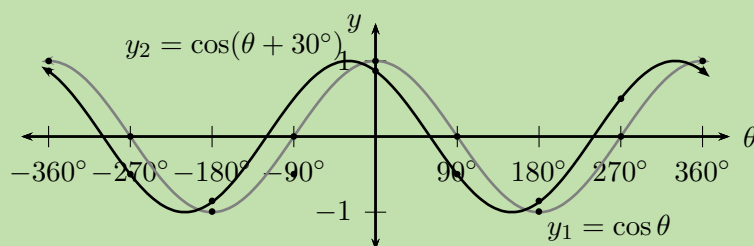
SOLUTION

Step 1: Examine the equations of the form $y = \cos(\theta + p)$

Notice that for $y_1 = \cos\theta$ we have $p = 0$ (no phase shift) and for $y_2 = \cos(\theta + 30°)$, $p < 0$ therefore the graph shifts to the left by $30°$.

Step 2: Complete a table of values

θ	$-360°$	$-270°$	$-180°$	$-90°$	$0°$	$90°$	$180°$	$270°$	$360°$
$\cos\theta$	1	0	-1	0	1	0	-1	0	1
$\cos(\theta+30°)$	$0{,}87$	$-0{,}5$	$-0{,}87$	$0{,}5$	$0{,}87$	$-0{,}5$	$-0{,}87$	$0{,}5$	$0{,}87$

Step 3: Sketch the cosine graphs

Step 4: Complete the table

	y_1	y_2
period	$360°$	$360°$
amplitude	1	1
domain	$[-360°; 360°]$	$[-360°; 360°]$
range	$[-1; 1]$	$[-1; 1]$
maximum turning points	$(-360°; 1)$, $(0°; 1)$ and $(360°; 1)$	$(-30°; 1)$ and $(330°; 1)$
minimum turning points	$(-180°; -1)$ and $(180°; -1)$	$(-210°; -1)$ and $(150°; -1)$
y**-intercept(s)**	$(0°; 0)$	$(0°; 0{,}87)$
x**-intercept(s)**	$(-270°; 0)$, $(-90°; 0)$, $(90°; 0)$ and $(270°; 0)$	$(-300°; 0)$, $(-120°; 0)$, $(60°; 0)$ and $(240°; 0)$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \cos(\theta + p)$:

Domain and range

The domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value for θ for which $f(\theta)$ is undefined.

The range is $\{f(\theta) : -1 \leq f(\theta) \leq 1, f(\theta) \in \mathbb{R}\}$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0°$ and solving for $f(\theta)$.

$$\begin{aligned} y &= \cos(\theta + p) \\ &= \cos(0° + p) \\ &= \cos p \end{aligned}$$

This gives the point $(0°; \cos p)$.

Exercise 5 – 26: Cosine functions of the form $y = \cos(\theta + p)$

Sketch the following functions for $-360° \leq \theta \leq 360°$.

For each function, determine the following:

- Period
- Amplitude
- Domain and range
- x- and y-intercepts
- Maximum and minimum turning points

1. $f(\theta) = \cos(\theta + 45°)$
2. $g(\theta) = \cos(\theta - 30°)$
3. $h(\theta) = \cos(\theta + 60°)$

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1. 22VP 2. 22VQ 3. 22VR

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Sketching cosine graphs EMBH7

Worked example 24: Sketching a cosine graph

QUESTION

Sketch the graph of $f(\theta) = \cos(180° - 3\theta)$ for $0° \leq \theta \leq 360°$.

SOLUTION

Step 1: Examine the form of the equation

Write the equation in the form $y = \cos k(\theta + p)$.

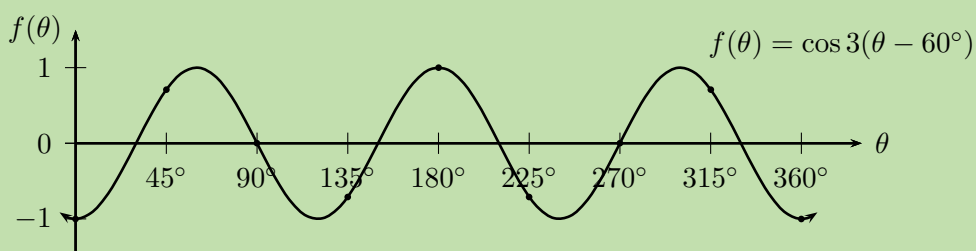
$$\begin{aligned} f(\theta) &= \cos(180° - 3\theta) \\ &= \cos(-3\theta + 180°) \\ &= \cos\left(-3(\theta - 60°)\right) \\ &= \cos 3(\theta - 60°) \end{aligned}$$

To draw a graph of the above function, the standard cosine graph, $y = \cos\theta$, must be changed in the following ways:

- decrease the period by a factor of 3
- shift to the right by $60°$.

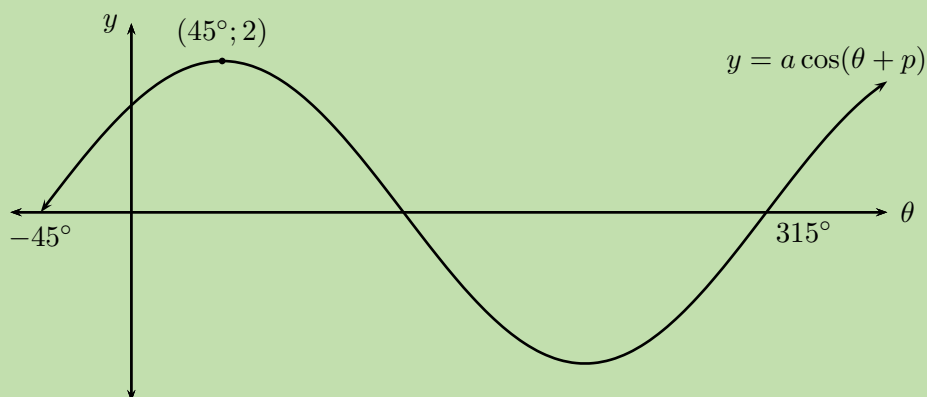
Step 2: Complete a table of values

θ	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$f(\theta)$	-1	$0{,}71$	0	$-0{,}71$	1	$-0{,}71$	0	$0{,}71$	-1

Step 3: Plot the points and join with a smooth curve

Period: $120°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[-1; 1]$
Maximum turning point: $(60°; 1)$, $(180°; 1)$ and $(300°; 1)$
Minimum turning point: $(0°; -1)$, $(120°; -1)$, $(240°; -1)$ and $(360°; -1)$
y-intercepts: $(0°; -1)$
x-intercept: $(30°; 0)$, $(90°; 0)$, $(150°; 0)$, $(210°; 0)$, $(270°; 0)$ and $(330°; 0)$

Worked example 25: Finding the equation of a cosine graph

QUESTION

Given the graph of $y = a\cos(k\theta + p)$, determine the values of a, k, p and the minimum turning point.

SOLUTION

Step 1: Determine the value of k

From the sketch we see that the period of the graph is $360°$, therefore $k = 1$.

$$y = a\cos(\theta + p)$$

Step 2: Determine the value of a

From the sketch we see that the maximum turning point is $(45°; 2)$, so we know that the amplitude of the graph is 2 and therefore $a = 2$.

$$y = 2\cos(\theta + p)$$

Step 3: Determine the value of p

Compare the given graph with the standard cosine function $y = \cos\theta$ and notice the difference in the maximum turning points. We see that the given function has been shifted to the right by $45°$, therefore $p = 45°$.

$$y = 2\cos(\theta - 45°)$$

Step 4: Determine the minimum turning point

At the minimum turning point, $y = -2$:

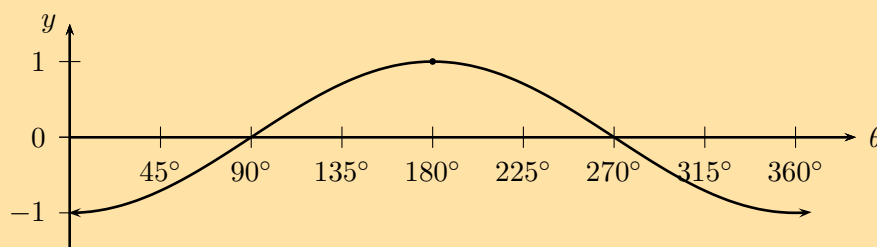
$$\begin{aligned} y &= 2\cos(\theta - 45°) \\ -2 &= 2\cos(\theta - 45°) \\ -1 &= \cos(\theta - 45°) \\ \cos^{-1}(-1) &= \theta - 45° \\ 180° &= \theta - 45° \\ 225° &= \theta \end{aligned}$$

This gives the point $(225°; -2)$.

Exercise 5 – 27: The cosine function

1. Sketch the following graphs on separate axes:

   a) $y = \cos(\theta + 15°)$ for $-180° \leq \theta \leq 180°$
   b) $f(\theta) = \frac{1}{3}\cos(\theta - 60°)$ for $-90° \leq \theta \leq 90°$
   c) $y = -2\cos\theta$ for $0° \leq \theta \leq 360°$
   d) $y = \cos(30° - \theta)$ for $-360° \leq \theta \leq 360°$
   e) $g(\theta) = 1 + \cos(\theta - 90°)$ for $0° \leq \theta \leq 360°$
   f) $y = \cos(2\theta + 60°)$ for $-360° \leq \theta \leq 360°$

2. Two girls are given the following graph:

   a) Audrey decides that the equation for the graph is a cosine function of the form $y = a\cos\theta$. Determine the value of a.
   b) Megan thinks that the equation for the graph is a cosine function of the form $y = \cos(\theta + p)$. Determine the value of p.
   c) What can they conclude?

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1a. 22VS 1b. 22VT 1c. 22VV 1d. 22VW 1e. 22VX 1f. 22VY
2. 22VZ

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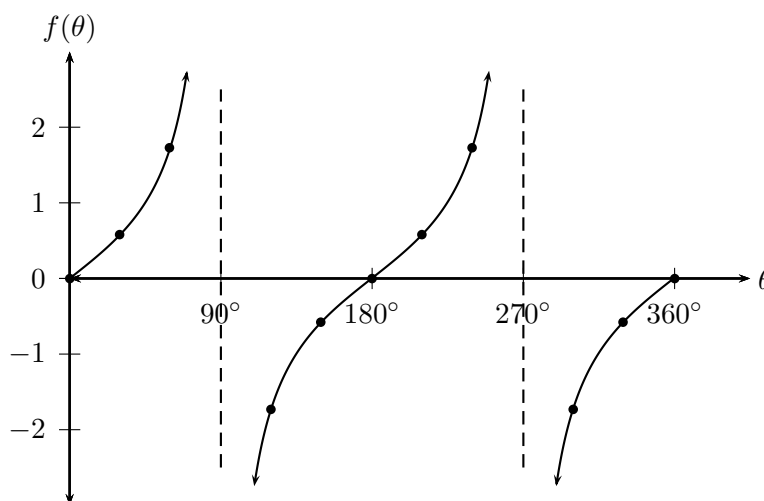
Revision EMBH9

Functions of the form $y = \tan\theta$ for $0° \leq \theta \leq 360°$

The dashed vertical lines are called the asymptotes. The asymptotes are at the values of θ where $\tan\theta$ is not defined.

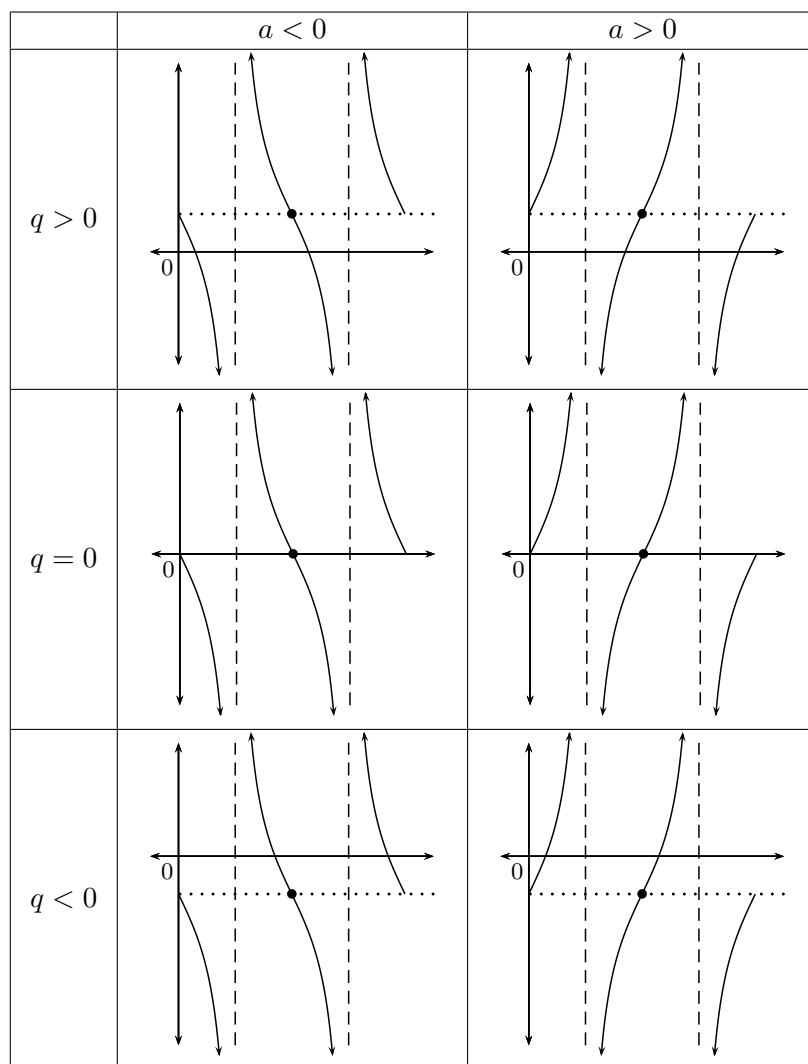
- Period: $180°$
- Domain: $\{\theta : 0° \leq \theta \leq 360°, \theta \neq 90°; 270°\}$
- Range: $\{f(\theta) : f(\theta) \in \mathbb{R}\}$
- x-intercepts: $(0°; 0)$, $(180°; 0)$, $(360°; 0)$
- y-intercept: $(0°; 0)$
- Asymptotes: the lines $\theta = 90°$ and $\theta = 270°$

Functions of the form $y = a\tan\theta + q$

Tangent functions of the general form $y = a\tan\theta + q$, where a and q are constants.

The effects of a and q on $f(\theta) = a\tan\theta + q$:

- **The effect of q on vertical shift**
  - For $q > 0$, $f(\theta)$ is shifted vertically upwards by q units.
  - For $q < 0$, $f(\theta)$ is shifted vertically downwards by q units.
- **The effect of a on shape**
  - For $a > 1$, branches of $f(\theta)$ are steeper.
  - For $0 < a < 1$, branches of $f(\theta)$ are less steep and curve more.

- For $a < 0$, there is a reflection about the x-axis.
- For $-1 < a < 0$, there is a reflection about the x-axis and the branches of the graph are less steep.
- For $a < -1$, there is a reflection about the x-axis and the branches of the graph are steeper.

Exercise 5 – 28: Revision

On separate axes, accurately draw each of the following functions for $0° \leq \theta \leq 360°$:

- Use tables of values if necessary.
- Use graph paper if available.

For each function determine the following:

- Period
- Domain and range

- x- and y-intercepts
- Asymptotes

1. $y_1 = \tan\theta - \frac{1}{2}$
2. $y_2 = -3\tan\theta$
3. $y_3 = \tan\theta + 2$
4. $y_4 = 2\tan\theta - 1$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22W2 2. 22W3 3. 22W4 4. 22W5

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Functions of the form $y = \tan(k\theta)$ EMBHB

Investigation: The effects of k on a tangent graph

1. Complete the following table for $y_1 = \tan\theta$ for $-360° \leq \theta \leq 360°$:

θ	$-360°$	$-300°$	$-240°$	$-180°$	$-120°$	$-60°$	$0°$
$\tan\theta$							
θ	$60°$	$120°$	$180°$	$240°$	$300°$	$360°$	
$\tan\theta$							

2. Use the table of values to plot the graph of $y_1 = \tan\theta$ for $-360° \leq \theta \leq 360°$.
3. On the same system of axes, plot the following graphs:
   a) $y_2 = \tan(-\theta)$
   b) $y_3 = \tan 3\theta$
   c) $y_4 = \tan\frac{\theta}{2}$
4. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4
period				
domain				
range				
y-intercept(s)				
x-intercept(s)				
asymptotes				
effect of k				

5. What do you notice about $y_1 = \tan\theta$ and $y_2 = \tan(-\theta)$?
6. Is $\tan(-\theta) = -\tan\theta$ a true statement? Explain your answer.
7. Can you deduce a formula for determining the period of $y = \tan k\theta$?

The effect of the parameter on $y = \tan k\theta$

The value of k affects the period of the tangent function. If k is negative, then the graph is reflected about the y-axis.

- **For** $k > 0$:

  For $k > 1$, the period of the tangent function decreases.

  For $0 < k < 1$, the period of the tangent function increases.

- **For** $k < 0$:

  For $-1 < k < 0$, the graph is reflected about the y-axis and the period increases.

  For $k < -1$, the graph is reflected about the y-axis and the period decreases.

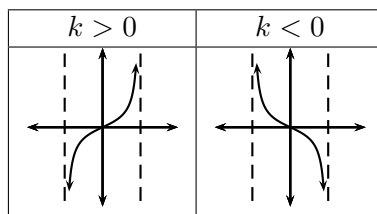
Negative angles:

$$\tan(-\theta) = -\tan\theta$$

Calculating the period:

To determine the period of $y = \tan k\theta$ we use,

$$\text{Period} = \frac{180^\circ}{|k|}$$

where $|k|$ is the absolute value of k.

Worked example 26: Tangent function

QUESTION

1. Sketch the following functions on the same set of axes for $-180° \leq \theta \leq 180°$.

   a) $y_1 = \tan\theta$

   b) $y_2 = \tan\frac{3\theta}{2}$

2. For each function determine the following:

   - Period
   - Domain and range
   - x- and y-intercepts
   - Asymptotes

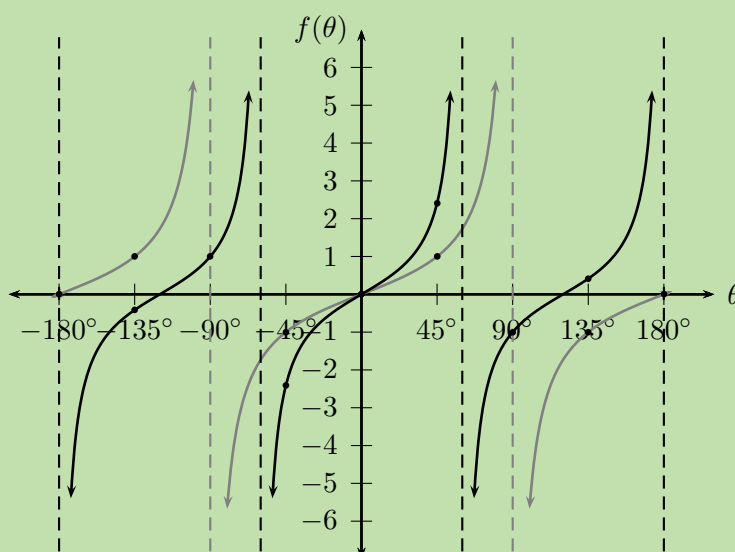
SOLUTION

Step 1: Examine the equations of the form $y = \tan k\theta$

Notice that $k > 1$ for $y_2 = \tan\frac{3\theta}{2}$, therefore the period of the graph decreases.

Step 2: Complete a table of values

θ	$-180°$	$-135°$	$-90°$	$-45°$	$0°$	$45°$	$90°$	$135°$	$180°$
$\tan\theta$	0	1	UNDEF	-1	0	1	UNDEF	-1	0
$\tan\frac{3\theta}{2}$	UNDEF	$-0{,}41$	1	$-2{,}41$	0	$2{,}41$	-1	$0{,}41$	UNDEF

Step 3: Sketch the tangent graphs

Step 4: Complete the table

	$y_1 = \tan\theta$	$y_2 = \tan\frac{3\theta}{2}$
period	$180°$	$120°$
domain	$\{\theta : -180° \leq \theta \leq 180°, \theta \neq -90°; 90°\}$	$\{\theta : -180° < \theta < 180°, \theta \neq -60°; 60°\}$
range	$\{f(\theta) : f(\theta) \in \mathbb{R}\}$	$\{f(\theta) : f(\theta) \in \mathbb{R}\}$
y-intercept(s)	$(0°; 0)$	$(0°; 0)$
x-intercept(s)	$(-180°; 0)$, $(0°; 0)$ and $(180°; 0)$	$(-120°; 0)$, $(0°; 0)$ and $(120°; 0)$
asymptotes	$\theta = -90°$ and $\theta = 90°$	$\theta = -180°$; $-60°$ and $180°$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \tan k\theta$:

Domain and range

The domain of one branch is $\{\theta : -\frac{90^\circ}{k} < \theta < \frac{90^\circ}{k}, \theta \in \mathbb{R}\}$ because $f(\theta)$ is undefined for $\theta = -\frac{90^\circ}{k}$ and $\theta = \frac{90^\circ}{k}$.

The range is $\{f(\theta) : f(\theta) \in \mathbb{R}\}$ or $(-\infty; \infty)$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0$ and solving for $f(\theta)$.

$$\begin{aligned} y &= \tan k\theta \\ &= \tan 0^\circ \\ &= 0 \end{aligned}$$

This gives the point $(0^\circ; 0)$.

Asymptotes

These are the values of $k\theta$ for which $\tan k\theta$ is undefined.

Exercise 5 – 29: Tangent functions of the form $y = \tan k\theta$

Sketch the following functions for $-180^\circ \leq \theta \leq 180^\circ$. For each graph determine:

- Period
- Domain and range
- x- and y-intercepts
- Asymptotes

1. $f(\theta) = \tan 2\theta$
2. $g(\theta) = \tan \frac{3\theta}{4}$
3. $h(\theta) = \tan(-2\theta)$
4. $k(\theta) = \tan \frac{2\theta}{3}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22W6 2. 22W7 3. 22W8 4. 22W9

Functions of the form $y = \tan(\theta + p)$ EMBHC

We now consider tangent functions of the form $y = \tan(\theta + p)$ and the effects of parameter p.

Investigation: The effects of p on a tangent graph

1. On the same system of axes, plot the following graphs for $-360° \leq \theta \leq 360°$:

   a) $y_1 = \tan\theta$

   b) $y_2 = \tan(\theta - 60°)$

   c) $y_3 = \tan(\theta - 90°)$

   d) $y_4 = \tan(\theta + 60°)$

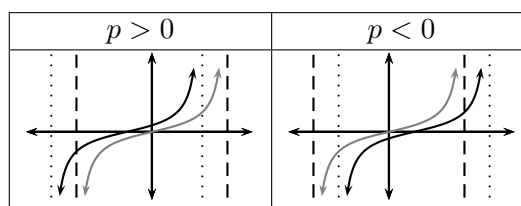
   e) $y_5 = \tan(\theta + 180°)$

2. Use your sketches of the functions above to complete the following table:

	y_1	y_2	y_3	y_4	y_5
period					
domain					
range					
y-intercept(s)					
x-intercept(s)					
asymptotes					
effect of p					

The effect of the parameter on $y = \tan(\theta + p)$

The effect of p on the tangent function is a horizontal shift (or phase shift); the entire graph slides to the left or to the right.

- For $p > 0$, the graph of the tangent function shifts to the left by p.
- For $p < 0$, the graph of the tangent function shifts to the right by p.

Worked example 27: Tangent function

QUESTION

1. Sketch the following functions on the same set of axes for $-180° \leq \theta \leq 180°$.

   a) $y_1 = \tan\theta$

   b) $y_2 = \tan(\theta + 30°)$

   For each function determine the following:

2. - Period
   - Domain and range
   - x- and y-intercepts
   - Asymptotes

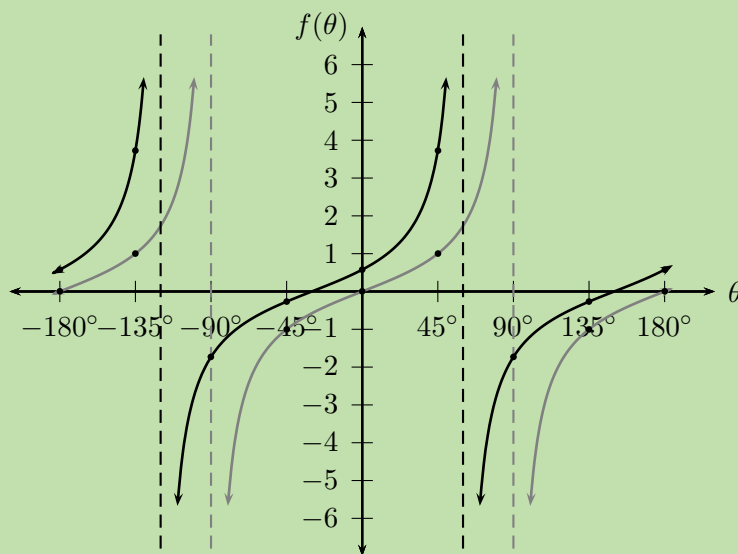
SOLUTION

Step 1: Examine the equations of the form $y = \tan(\theta + p)$

Notice that for $y_1 = \tan\theta$ we have $p = 0°$ (no phase shift) and for $y_2 = \tan(\theta + 30°)$, $p > 0$ therefore the graph shifts to the left by $30°$.

Step 2: Complete a table of values

θ	$-180°$	$-135°$	$-90°$	$-45°$	$0°$	$45°$	$90°$	$135°$	$180°$
$\tan\theta$	0	1	UNDEF	-1	0	1	UNDEF	-1	0
$\tan(\theta+30°)$	$0{,}58$	$3{,}73$	$-1{,}73$	$-0{,}27$	$0{,}58$	$3{,}73$	$-1{,}73$	$-0{,}27$	$0{,}58$

Step 3: Sketch the tangent graphs

Step 4: Complete the table

	$y_1 = \tan\theta$	$y_2 = \tan(\theta + 30°)$
period	$180°$	$180°$
domain	$\{\theta : -180° \leq \theta \leq 180°, \theta \neq -90°; 90°\}$	$\{\theta : -180° \leq \theta \leq 180°, \theta \neq -120°; 60°\}$
range	$(-\infty; \infty)$	$(-\infty; \infty)$
y-intercept(s)	$(0°; 0)$	$(0°; 0{,}58)$
x-intercept(s)	$(-180°; 0)$, $(0°; 0)$ and $(180°; 0)$	$(-30°; 0)$ and $(150°; 0)$
asymptotes	$\theta = -90°$ and $\theta = 90°$	$\theta = -120°$ and $\theta = 60°$

Discovering the characteristics

For functions of the general form: $f(\theta) = y = \tan(\theta + p)$:

Domain and range

The domain of one branch is $\{\theta : \theta \in (-90° - p; 90° - p)\}$ because the function is undefined for $\theta = -90° - p$ and $\theta = 90° - p$.

The range is $\{f(\theta) : f(\theta) \in \mathbb{R}\}$.

Intercepts

The x-intercepts are determined by letting $f(\theta) = 0$ and solving for θ.

The y-intercept is calculated by letting $\theta = 0°$ and solving for $f(\theta)$.

$$\begin{aligned} y &= \tan(\theta + p) \\ &= \tan(0° + p) \\ &= \tan p \end{aligned}$$

This gives the point $(0°; \tan p)$.

Exercise 5 – 30: Tangent functions of the form $y = \tan(\theta + p)$

Sketch the following functions for $-360° \leq \theta \leq 360°$.

For each function, determine the following:

- Period
- Domain and range
- x- and y-intercepts
- Asymptotes

1. $f(\theta) = \tan(\theta + 45°)$
2. $g(\theta) = \tan(\theta - 30°)$
3. $h(\theta) = \tan(\theta + 60°)$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22WB 2. 22WC 3. 22WD

Worked example 28: Sketching a tangent graph

QUESTION

Sketch the graph of $f(\theta) = \tan \frac{1}{2}(\theta - 30°)$ for $-180° \leq \theta \leq 180°$.

SOLUTION

Step 1: Examine the form of the equation

From the equation we see that $0 < k < 1$, therefore the branches of the graph will be less steep than the standard tangent graph $y = \tan \theta$. We also notice that $p < 0$ so the graph will be shifted to the right on the x-axis.

Step 2: Determine the period

The period for $f(\theta) = \tan \frac{1}{2}(\theta - 30°)$ is:

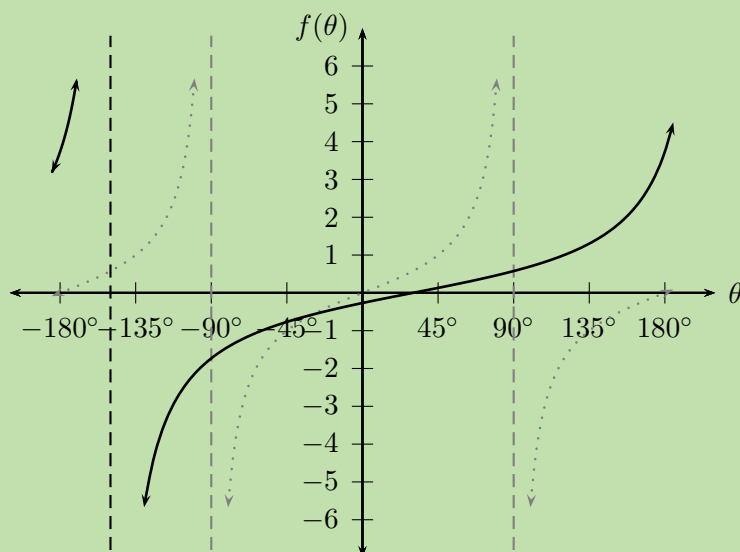
$$\begin{aligned} \text{Period} &= \frac{180°}{|k|} \\ &= \frac{180°}{\frac{1}{2}} \\ &= 360° \end{aligned}$$

Step 3: Determine the asymptotes

The standard tangent graph, $y = \tan \theta$, for $-180° \leq \theta \leq 180°$ is undefined at $\theta = -90°$ and $\theta = 90°$. Therefore we can determine the asymptotes of $f(\theta) = \tan \frac{1}{2}(\theta - 30°)$:

- $\frac{-90°}{0{,}5} + 30° = -150°$
- $\frac{90°}{0{,}5} + 30° = 210°$

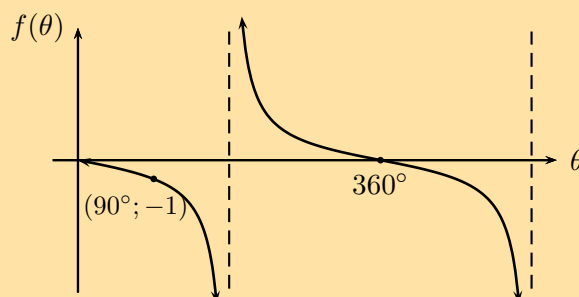
The asymptote at $\theta = 210°$ lies outside the required interval.

Step 4: Plot the points and join with a smooth curve

Period: $360°$
Domain: $\{\theta : -180° \leq \theta \leq 180°, \theta \neq -150°\}$
Range: $(-\infty; \infty)$
y-intercepts: $(0°; -0{,}27)$
x-intercept: $(30°; 0)$
Asymptotes: $\theta = -150°$

Exercise 5 – 31: The tangent function

1. Sketch the following graphs on separate axes:

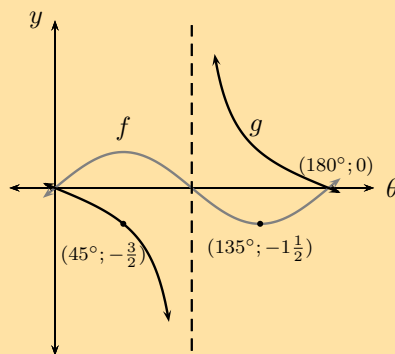
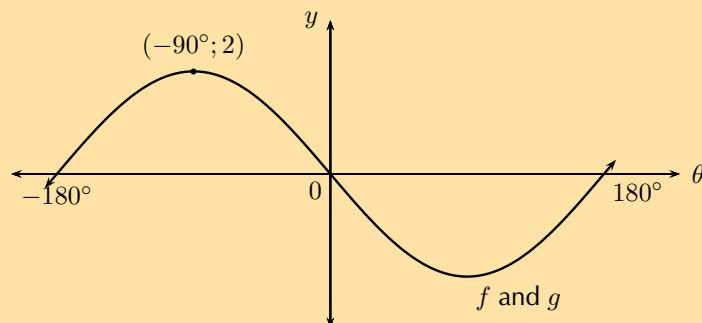
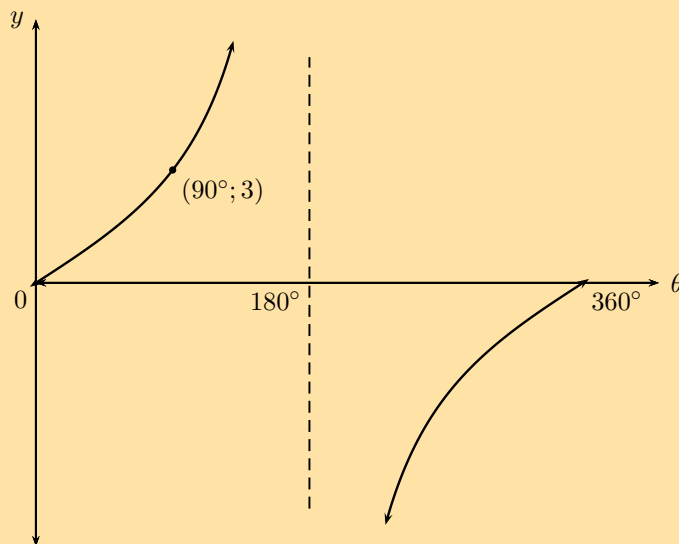
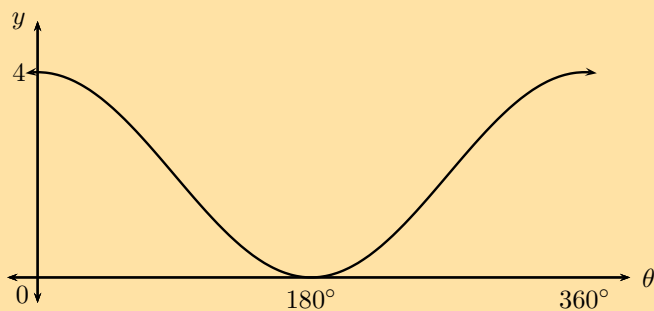
   a) $y = \tan\theta - 1$ for $-90° \leq \theta \leq 90°$
   b) $f(\theta) = -\tan 2\theta$ for $0° \leq \theta \leq 90°$
   c) $y = \frac{1}{2}\tan(\theta + 45°)$ for $0° \leq \theta \leq 360°$
   d) $y = \tan(30° - \theta)$ for $-180° \leq \theta \leq 180°$

2. Given the graph of $y = a\tan k\theta$, determine the values of a and k.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

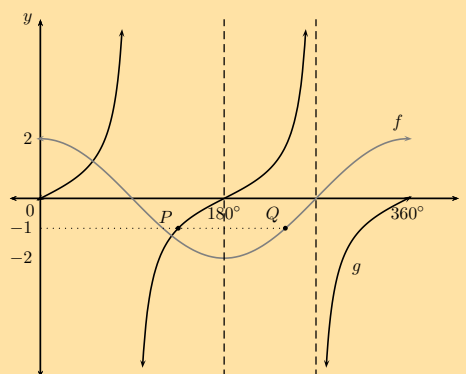
1a. 22WF 1b. 22WG 1c. 22WH 1d. 22WJ 2. 22WK

Exercise 5 – 32: Mixed exercises

1. Determine the equation for each of the following:

a) $f(\theta) = a \sin k\theta$ and $g(\theta) = a \tan \theta$

b) $f(\theta) = a \sin k\theta$ and $g(\theta) = a \cos(\theta + p)$

c) $y = a \tan k\theta$

d) $y = a \cos \theta + q$

2. Given the functions $f(\theta) = 2\sin\theta$ and $g(\theta) = \cos\theta + 1$:

   a) Sketch the graphs of both functions on the same system of axes, for $0° \leq \theta \leq 360°$. Indicate the turning points and intercepts on the diagram.
   b) What is the period of f?
   c) What is the amplitude of g?
   d) Use your sketch to determine how many solutions there are for the equation $2\sin\theta - \cos\theta = 1$. Give one of the solutions.
   e) Indicate on your sketch where on the graph the solution to $2\sin\theta = -1$ is found.

3. The sketch shows the two functions $f(\theta) = a\cos\theta$ and $g(\theta) = \tan\theta$ for $0° \leq \theta \leq 360°$. Points $P(135°; b)$ and $Q(c; -1)$ lie on $g(\theta)$ and $f(\theta)$ respectively.

   a) Determine the values of a, b and c.
   b) What is the period of g?
   c) Solve the equation $\cos\theta = \frac{1}{2}$ graphically and show your answer(s) on the diagram.
   d) Determine the equation of the new graph if g is reflected about the x-axis and shifted to the right by $45°$.

4. Sketch the graphs of $y_1 = -\frac{1}{2}\sin(\theta + 30°)$ and $y_2 = \cos(\theta - 60°)$, on the same system of axes for $0° \leq \theta \leq 360°$.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22WM 1b. 22WN 1c. 22WP 1d. 22WQ 2. 22WR 3. 22WS 4. 22WT

5.8 Summary EMBHF

See presentation: 22WV at www.everythingmaths.co.za

1. Parabolic functions:

   Standard form: $y = ax^2 + bx + c$

   - y-intercept: $(0; c)$
   - x-intercept: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
   - Turning point: $\left(-\frac{b}{2a}; -\frac{b^2}{4a} + c\right)$
   - Axis of symmetry: $x = -\frac{b}{2a}$

   Completed square form: $y = a(x+p)^2 + q$

   - Turning point: $(-p; q)$
   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right
   - $q > 0$: vertical shift up
   - $q < 0$: vertical shift down

2. Average gradient:

   - Average gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

3. Hyperbolic functions:

   Standard form: $y = \frac{k}{x}$

   - $k > 0$: first and third quadrant
   - $k < 0$: second and fourth quadrant

   Shifted form: $y = \frac{k}{x+p} + q$

   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right
   - $q > 0$: vertical shift up
   - $q < 0$: vertical shift down
   - Asymptotes: $x = -p$ and $y = q$

4. Exponential functions:

   Standard form: $y = ab^x$

   - $a > 0$: above x-axis
   - $a < 0$: below x-axis
   - $b > 1$: increasing function if $a > 0$; decreasing function if $a < 0$
   - $0 < b < 1$: decreasing function if $a > 0$; increasing function if $a < 0$

   Shifted form: $y = ab^{(x+p)} + q$

   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right

- $q > 0$: vertical shift up
- $q < 0$: vertical shift down
- Asymptotes: $y = q$

5. Sine functions:

   Shifted form: $y = a\sin(k\theta + p) + q$

   - Period $= \frac{360^\circ}{|k|}$
   - $k > 1$ or $k < -1$: period decreases
   - $0 < k < 1$ or $-1 < k < 0$: period increases
   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right
   - $q > 0$: vertical shift up
   - $q < 0$: vertical shift down
   - $\sin(-\theta) = -\sin\theta$

6. Cosine functions:

   Shifted form: $y = a\cos(k\theta + p) + q$

   - Period $= \frac{360^\circ}{|k|}$
   - $k > 1$ or $k < -1$: period decreases
   - $0 < k < 1$ or $-1 < k < 0$: period increases
   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right
   - $q > 0$: vertical shift up
   - $q < 0$: vertical shift down
   - $\cos(-\theta) = \cos\theta$

7. Tangent functions:

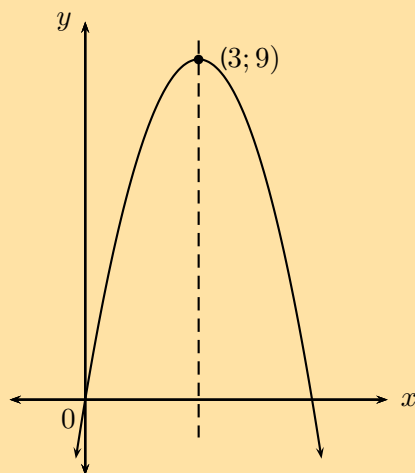
   Shifted form: $y = a\tan(k\theta + p) + q$

   - Period $= \frac{180^\circ}{|k|}$
   - $k > 1$ or $k < -1$: period decreases
   - $0 < k < 1$ or $-1 < k < 0$: period increases
   - $p > 0$: horizontal shift left
   - $p < 0$: horizontal shift right
   - $q > 0$: vertical shift up
   - $q < 0$: vertical shift down
   - $\tan(-\theta) = -\tan\theta$
   - Asymptotes: $\frac{90^\circ - p}{k} \pm \frac{180^\circ n}{k}$, $n \in \mathbb{Z}$

Exercise 5 – 33: End of chapter exercises

1. Show that if $a < 0$, then the range of $f(x) = a(x+p)^2 + q$ is $\{f(x) : f(x) \in (-\infty, q]\}$.

2. If $(2;7)$ is the turning point of $f(x) = -2x^2 - 4ax + k$, find the values of the constants a and k.

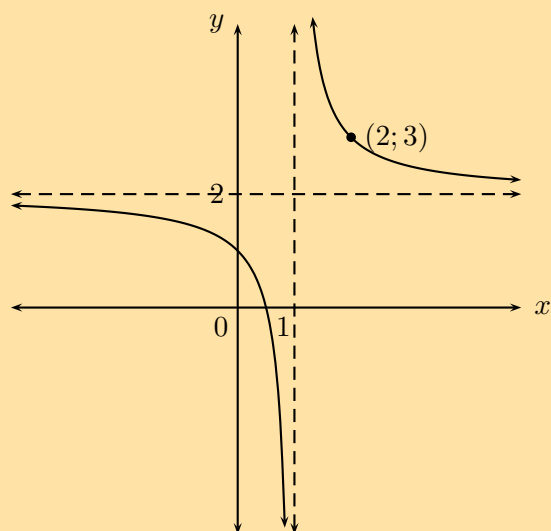
3. The following graph is represented by the equation $f(x) = ax^2 + bx$. The coordinates of the turning point are $(3;9)$. Show that $a = -1$ and $b = 6$.

4. Given: $f(x) = x^2 - 2x + 3$. Give the equation of the new graph originating if:

   a) the graph of f is moved three units to the left.

   b) the x-axis is moved down three units.

5. A parabola with turning point $(-1;-4)$ is shifted vertically by 4 units upwards. What are the coordinates of the turning point of the shifted parabola?

6. Plot the graph of the hyperbola defined by $y = \frac{2}{x}$ for $-4 \leq x \leq 4$. Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then?

7. Based on the graph of $y = \frac{k}{(x+p)} + q$, determine the equation of the graph with asymptotes $y = 2$ and $x = 1$ and passing through the point $(2;3)$.

8. The columns in the table below give the y-values for the following functions: $y = a^x$, $y = a^{x+1}$ and $y = a^x + 1$. Match each function to the correct column.

x	A	B	C
-2	7,25	6,25	2,5
-1	3,5	2,5	1
0	2	1	0,4
1	1,4	0,4	0,16
2	1,16	0,16	0,064

9. The graph of $f(x) = 1 + a \cdot 2^x$ (a is a constant) passes through the origin.

   a) Determine the value of a.

   b) Determine the value of $f(-15)$ correct to five decimal places.

   c) Determine the value of x, if $P(x; 0{,}5)$ lies on the graph of f.

   d) If the graph of f is shifted 2 units to the right to give the function h, write down the equation of h.

10. The graph of $f(x) = a \cdot b^x$ ($a \neq 0$) has the point $P(2; 144)$ on f.

    a) If $b = 0{,}75$, calculate the value of a.

    b) Hence write down the equation of f.

    c) Determine, correct to two decimal places, the value of $f(13)$.

    d) Describe the transformation of the curve of f to h if $h(x) = f(-x)$.

11. Using your knowledge of the effects of p and k draw a rough sketch of the following graphs without a table of values.

    a) $y = \sin 3\theta$ for $-180° \leq \theta \leq 180°$

    b) $y = -\cos 2\theta$ for $0° \leq \theta \leq 180°$

    c) $y = \tan \frac{1}{2}\theta$ for $0° \leq \theta \leq 360°$

    d) $y = \sin(\theta - 45°)$ for $-360° \leq \theta \leq 360°$

    e) $y = \cos(\theta + 45°)$ for $0° \leq \theta \leq 360°$

    f) $y = \tan(\theta - 45°)$ for $0° \leq \theta \leq 360°$

    g) $y = 2\sin 2\theta$ for $-180° \leq \theta \leq 180°$

    h) $y = \sin(\theta + 30°) + 1$ for $-360° \leq \theta \leq 0°$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22WW	2. 22WX	3. 22WY	4. 22WZ	5. 22X2	6. 22X3
7. 22X4	8. 22X5	9. 22X6	10. 22X7	11a. 22X8	11b. 22X9
11c. 22XB	11d. 22XC	11e. 22XD	11f. 22XF	11g. 22XG	11h. 22XH

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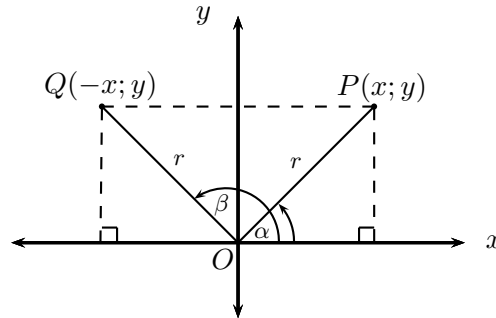
CHAPTER

Trigonometry

6.1 Revision EMBHG

Trigonometric ratios

We plot the points $P(x;y)$ and $Q(-x;y)$ in the Cartesian plane and measure the angles from the positive x-axis to the terminal arms (OP and OQ).

$P(x;y)$ lies in the first quadrant with $P\hat{O}X = \alpha$ and $Q(-x;y)$ lies in the second quadrant with $Q\hat{O}X = \beta$.

Using the theorem of Pythagoras we have that

$$\begin{aligned} OP^2 &= x^2 + y^2 \\ \text{And } OQ^2 &= (-x)^2 + y^2 \\ &= x^2 + y^2 \\ \therefore OP &= OQ \end{aligned}$$

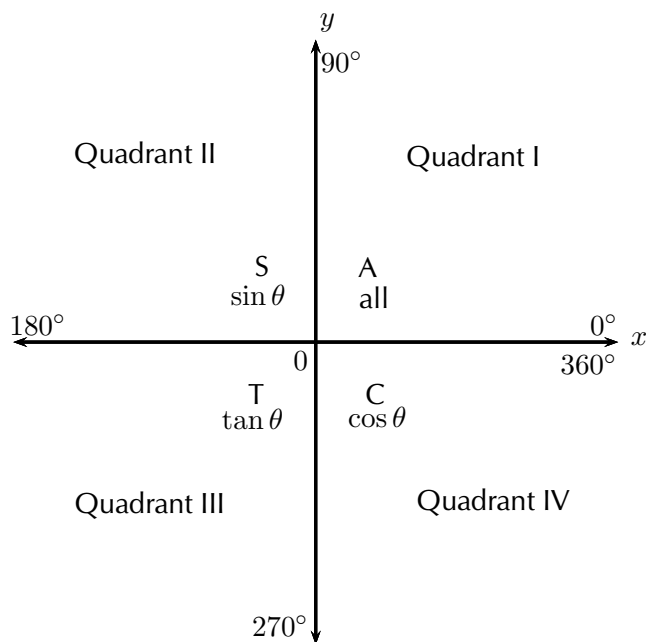
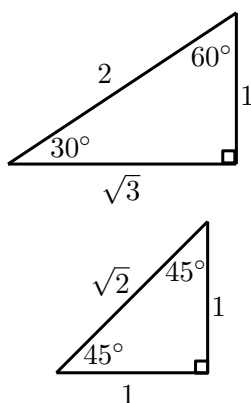
Let $OP = OQ = r$.

Trigonometric ratios

$$\begin{aligned} \sin\alpha &= \frac{y}{r} \\ \cos\alpha &= \frac{x}{r} \\ \tan\alpha &= \frac{y}{x} \end{aligned}$$

In the second quadrant we notice that $-x < 0$

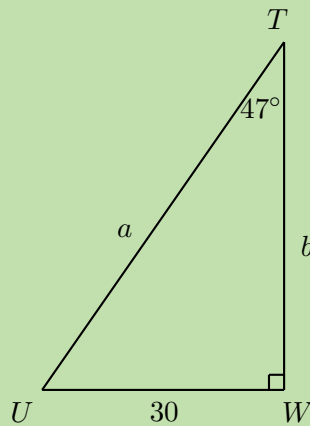
$$\begin{aligned} \sin\beta &= \frac{y}{r} \\ \cos\beta &= -\frac{x}{r} \\ \tan\beta &= -\frac{y}{x} \end{aligned}$$

Similarily, in the third and fourth quadrants the sign of the trigonometric ratios depends on the signs of x and y:

Special angles

θ	0°	30°	45°	60°	90°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

See video: 22XJ at www.everythingmaths.co.za

Worked example 1: Solving equations

QUESTION

Determine the values of a and b in the right-angled triangle TUW (correct to one decimal place):

SOLUTION

Step 1: Identify the opposite and adjacent sides and the hypotenuse

Step 2: Determine the value of a

$$\begin{aligned}\sin\theta &= \frac{\text{opposite side}}{\text{hypotenuse}}\\ \sin 47° &= \frac{30}{a}\\ a &= \frac{30}{\sin 47°}\\ \therefore a &= 41{,}0\end{aligned}$$

Step 3: Determine the value of b

Always try to use the information that is given for calculations and not answers that you have worked out in case you have made an error. For example, avoid using $a = 41{,}0$ to determine the value of b.

$$\begin{aligned}\tan\theta &= \frac{\text{opposite side}}{\text{adjacent side}}\\ \tan 47° &= \frac{30}{b}\\ b &= \frac{30}{\tan 47°}\\ \therefore b &= 28{,}0\end{aligned}$$

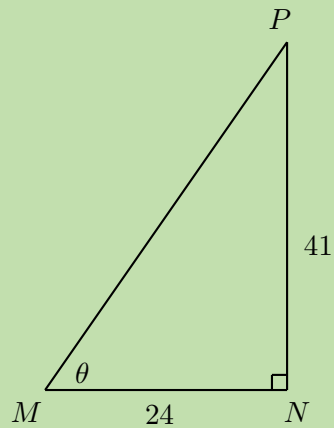
Step 4: Write the final answer

$a = 41{,}0$ units and $b = 28{,}0$ units.

Finding an angle

Worked example 2: Finding an angle

QUESTION

Calculate the value of θ in the right-angled triangle MNP (correct to one decimal place):

SOLUTION

Step 1: Identify the opposite and adjacent sides and the hypotenuse

Step 2: Determine the value of θ

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\tan\theta = \frac{41}{24}$$

$$\therefore \theta = \tan^{-1}\left(\frac{41}{24}\right)$$

$$\theta = 59{,}7^{\circ}$$

Worked example 3: Finding an angle

QUESTION

Given $2\sin\frac{\theta}{2} = \cos 43°$, for $\theta \in [0°; 90°]$, determine the value of θ (correct to one decimal place).

SOLUTION

Step 1: Simplify the equation

Avoiding rounding off in calculations until you have determined the final answer. In the calculation below, the dots indicate that the number has not been rounded so that the answer is as accurate as possible.

$$\begin{aligned} 2\sin\frac{\theta}{2} &= \cos 43° \\ \sin\frac{\theta}{2} &= \frac{\cos 43°}{2} \\ \frac{\theta}{2} &= \sin^{-1}(0{,}365\ldots) \\ \theta &= 2(21{,}449\ldots) \\ \therefore \theta &= 42{,}9° \end{aligned}$$

Two-dimensional problems

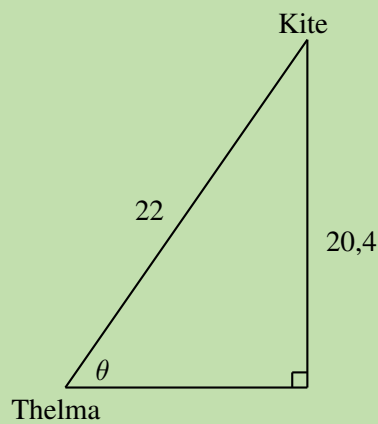
Worked example 4: Flying a kite

QUESTION

Thelma flies a kite on a 22 m piece of string and the height of the kite above the ground is 20,4 m. Determine the angle of inclination of the string (correct to one decimal place).

SOLUTION

Step 1: Draw a sketch and identify the opposite and adjacent sides and the hypotenuse

Let the angle of inclination of the string be θ.

Step 2: Use an appropriate trigonometric ratio to find θ

$$\begin{aligned}\sin\theta &= \frac{\text{opposite side}}{\text{hypotenuse}}\\ &= \frac{20{,}4}{22}\\ \theta &= \sin^{-1}(0{,}927\ldots)\\ \therefore \theta &= 68{,}0^\circ\end{aligned}$$

Exercise 6 – 1: Revision

1. If $p = 49^\circ$ and $q = 32^\circ$, use a calculator to determine whether the following statements are true of false:

   a) $\sin p + 3\sin p = 4\sin p$

   b) $\dfrac{\sin q}{\cos q} = \tan q$

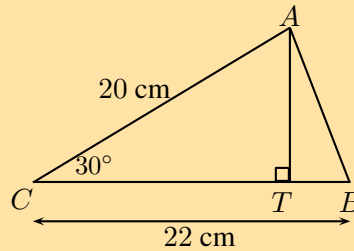
   c) $\cos(p - q) = \cos p - \cos q$

   d) $\sin(2p) = 2\sin p\cos p$

2. Determine the following angles (correct to one decimal place):

   a) $\cos\alpha = 0{,}64$
   b) $\sin\theta + 2 = 2{,}65$
   c) $\dfrac{1}{2}\cos 2\beta = 0{,}3$
   d) $\tan\frac{\theta}{3} = \sin 48^\circ$
   e) $\cos 3p = 1{,}03$
   f) $2\sin 3\beta + 1 = 2{,}6$
   g) $\dfrac{\sin\theta}{\cos\theta} = 4\frac{2}{3}$

3. In $\triangle ABC, A\hat{C}B = 30^\circ, AC = 20$ cm and $BC = 22$ cm. The perpendicular line from A intersects BC at T.

Determine:

a) the length TC

b) the length AT

c) the angle $B\hat{A}T$

4. A rhombus has a perimeter of 40 cm and one of the internal angles is 30°.

   a) Determine the length of the sides.

   b) Determine the lengths of the diagonals.

   c) Calculate the area of the rhombus.

5. Simplify the following without using a calculator:

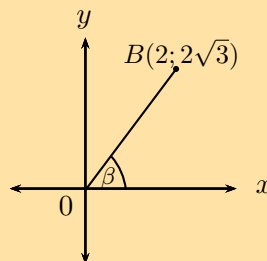
   a) $2\sin 45° \times 2\cos 45°$

   b) $\cos^2 30° - \sin^2 60°$

   c) $\sin 60° \cos 30° - \cos 60° \sin 30° - \tan 45°$

   d) $4\sin 60° \cos 30° - 2\tan 45° + \tan 60° - 2\sin 60°$

   e) $\sin 60° \times \sqrt{2\tan 45° + 1} - \sin 30°$

6. Given the diagram below.

   Determine the following without using a calculator:

   a) β

   b) $\cos\beta$

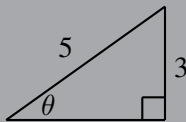
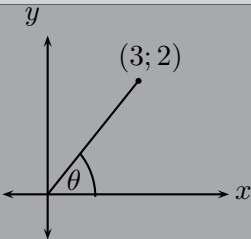
   c) $\cos^2\beta + \sin^2\beta$

7. The 10 m ladder of a fire truck leans against the wall of a burning building at an angle of 60°. The height of an open window is 9 m from the ground. Will the ladder reach the window?

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 22XK 2a. 22XM 2b. 22XN 2c. 22XP 2d. 22XQ 2e. 22XR
2f. 22XS 2g. 22XT 3. 22XV 4. 22XW 5a. 22XX 5b. 22XY
5c. 22XZ 5d. 22Y2 5e. 22Y3 6. 22Y4 7. 22Y5

6.2 Trigonometric identities

EMBHH

An identity is a mathematical statement that equates one quantity with another. Trigonometric identities allow us to simplify a given expression so that it contains sine and cosine ratios only. This enables us to solve equations and also to prove other identities.

Quotient identity

Investigation: Quotient identity

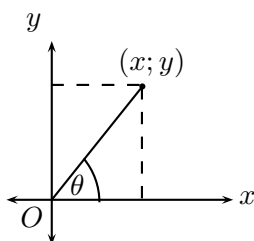
1. Complete the table without using a calculator, leaving your answer in surd form where applicable:

	$\theta = 45°$		
$\sin\theta$			
$\cos\theta$			
$\frac{\sin\theta}{\cos\theta}$			
$\tan\theta$			

2. Examine the last two rows of the table and make a conjecture.
3. Are there any values of θ for which your conjecture would not be true? Explain your answer.

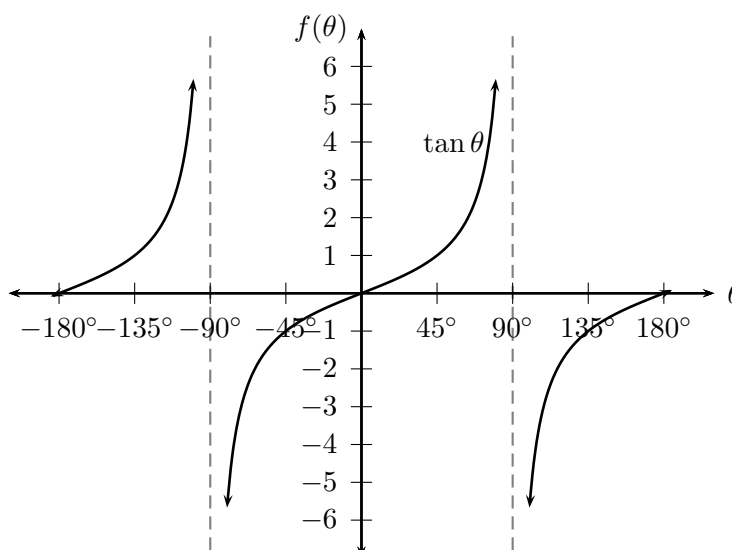
We know that $\tan\theta$ is defined as:

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Using the diagram below and the theorem of Pythagoras, we can write the tangent function in terms of x, y and r:

$$\begin{aligned}\tan\theta &= \frac{y}{x} \\ &= \frac{y}{x} \times \frac{r}{r} \\ &= \frac{y}{r} \times \frac{r}{x} \\ &= \frac{y}{r} \div \frac{x}{r} \\ &= \sin\theta \div \cos\theta \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

This is the quotient identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

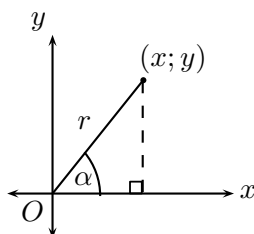
Notice that $\tan\theta$ is undefined if $\cos\theta = 0$, therefore $\theta \neq k \times 90°$, where k is an odd integer.

Square identity

Investigation: Square identity

1. Use a calculator to complete the following table:

$\sin^2 80° + \cos^2 80° =$	
$\cos^2 23° + \sin^2 23° =$	
$\sin 50° + \cos 50° =$	
$\sin^2 67° - \cos^2 67° =$	
$\sin^2 67° + \cos^2 67° =$	

2. What do you notice? Make a conjecture.

3. Draw a sketch and prove your conjecture in general terms, using x, y and r.

Using the theorem of Pythagoras, we can write the sine and cosine functions in terms of x, y and r:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ &= \frac{y^2 + x^2}{r^2} \\ &= \frac{r^2}{r^2} \\ &= 1\end{aligned}$$

This is the square identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Other forms of the square identity

Complete the following:

1. $\sin^2\theta = 1 - \ldots\ldots$
2. $\cos\theta = \pm\sqrt{\ldots\ldots}$
3. $\sin^2\theta = (1 + \ldots\ldots)(1 - \ldots\ldots)$
4. $\cos^2\theta - 1 = \ldots\ldots$

Here are some useful tips for proving identities:

- Change all trigonometric ratios to sine and cosine.
- Choose one side of the equation to simplify and show that it is equal to the other side.
- Usually it is better to choose the more complicated side to simplify.
- Sometimes we need to simplify both sides of the equation to show that they are equal.
- A square root sign often indicates that we need to use the square identity.
- We can also add to the expression to make simplifying easier:
  - replace 1 with $\sin^2\theta + \cos^2\theta$.
  - multiply by 1 in the form of a suitable fraction, for example $\dfrac{1+\sin\theta}{1+\sin\theta}$.

See video: 22Y6 at www.everythingmaths.co.za

Worked example 5: Trigonometric identities

QUESTION

Simplify the following:

1. $\tan^2\theta \times \cos^2\theta$
2. $\dfrac{1}{\cos^2\theta} - \tan^2\theta$

SOLUTION

Step 1: Write the expression in terms of sine and cosine only

We use the square and quotient identities to write the given expression in terms of sine and cosine and then simplify as far as possible.

1.

$$\begin{aligned}\tan^2\theta \times \cos^2\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)^2 \times \cos^2\theta \\ &= \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \\ &= \sin^2\theta\end{aligned}$$

2.

$$\begin{aligned}\frac{1}{\cos^2\theta} - \tan^2\theta &= \frac{1}{\cos^2\theta} - \left(\frac{\sin\theta}{\cos\theta}\right)^2 \\ &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{1 - \sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta}{\cos^2\theta} \\ &= 1\end{aligned}$$

Worked example 6: Trigonometric identities

QUESTION

Prove: $\frac{1-\sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{1+\sin\alpha}$

SOLUTION

Step 1: Note restrictions

When working with fractions, we must be careful that the denominator does not equal 0. Therefore $\cos\theta \neq 0$ for the fraction on the left-hand side and $\sin\theta + 1 \neq 0$ for the fraction on the right-hand side.

Step 2: Simplify the left-hand side

This is not an equation that needs to be solved. We are required to show that one side of the equation is equal to the other. We can choose either of the two sides to simplify.

$$\begin{aligned}\text{LHS} &= \frac{1-\sin\alpha}{\cos\alpha} \\ &= \frac{1-\sin\alpha}{\cos\alpha} \times \frac{1+\sin\alpha}{1+\sin\alpha}\end{aligned}$$

Notice that we have not changed the equation — this is the same as multiplying by 1 since the numerator and the denominator are the same.

Step 3: Determine the lowest common denominator and simplify

$$\begin{aligned}\text{LHS} &= \frac{1-\sin^2\alpha}{\cos\alpha(1+\sin\alpha)} \\ &= \frac{\cos^2\alpha}{\cos\alpha(1+\sin\alpha)} \\ &= \frac{\cos\alpha}{1+\sin\alpha} \\ &= \text{RHS}\end{aligned}$$

Exercise 6 – 2: Trigonometric identities

1. Reduce the following to one trigonometric ratio:

   a) $\frac{\sin\alpha}{\tan\alpha}$

   b) $\cos^2\theta\tan^2\theta + \tan^2\theta\sin^2\theta$

   c) $1 - \sin\theta\cos\theta\tan\theta$

   d) $\left(\frac{1-\cos^2\beta}{\cos^2\beta}\right) - \tan^2\beta$

2. Prove the following identities and state restrictions where appropriate:

   a) $\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}$

   b) $\sin^2\alpha + (\cos\alpha - \tan\alpha)(\cos\alpha + \tan\alpha) = 1 - \tan^2\alpha$

   c) $\frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1} = \cos\theta$

   d) $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta} = \sin\theta + \cos\theta - \frac{1}{\sin\theta+\cos\theta}$

   e) $\left(\frac{\cos\beta}{\sin\beta} + \tan\beta\right)\cos\beta = \frac{1}{\sin\beta}$

   f) $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = d\frac{2\tan\theta}{\sin\theta\cos\theta}$

   g) $\frac{(1+\tan^2\alpha)\cos\alpha}{(1-\tan\alpha)} = \frac{1}{\cos\alpha - \sin\alpha}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22Y7 1b. 22Y8 1c. 22Y9 1d. 22YB 2a. 22YC 2b. 22YD
2c. 22YF 2d. 22YG 2e. 22YH 2f. 22YJ 2g. 22YK

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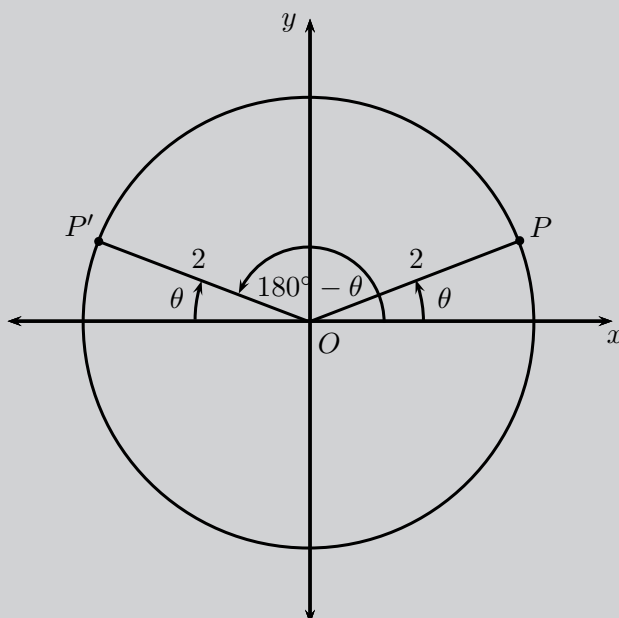
6.3 Reduction formula EMBHJ

Any trigonometric function whose argument is $90° \pm \theta$; $180° \pm \theta$ and $360° \pm \theta$ can be written simply in terms of θ.

Deriving reduction formulae EMBHK

Investigation: Reduction formulae for function values of 180° $\pm\,\theta$

1. **Function values of 180°** $-\,\theta$

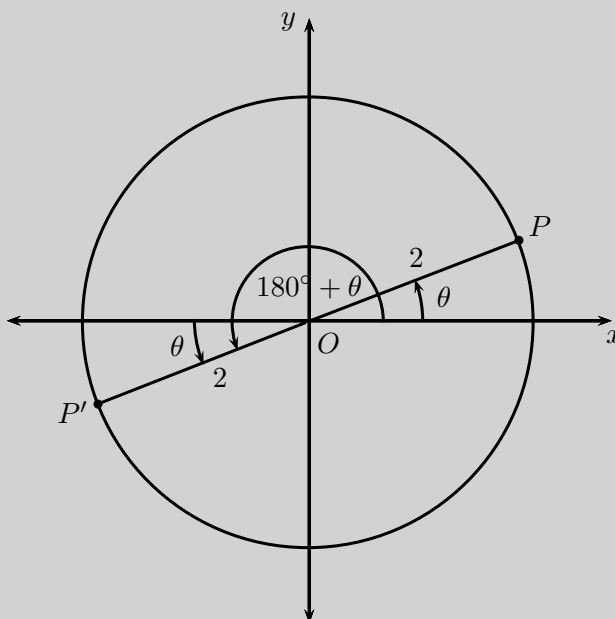
   In the figure $P(\sqrt{3};1)$ and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis.

   a) If points P and P' are symmetrical about the y-axis, determine the coordinates of P'.

   b) Write down values for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

   c) Use the coordinates for P' to determine $\sin(180° - \theta)$, $\cos(180° - \theta)$, $\tan(180° - \theta)$.

   d) From your results determine a relationship between the trigonometric function values of $(180° - \theta)$ and θ.

2. **Function values of 180° $+ \theta$**

   In the figure $P(\sqrt{3}; 1)$ and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis.

   a) If points P and P' are symmetrical about the origin (the two points are symmetrical about both the x-axis and the y-axis), determine the coordinates of P'.

   b) Use the coordinates for P' to determine $\sin(180° + \theta)$, $\cos(180° + \theta)$ and $\tan(180° + \theta)$.

   c) From your results determine a relationship between the trigonometric function values of $(180° + \theta)$ and θ.

3. Complete the following reduction formulae:

   a) $\sin(180° - \theta) = \ldots\ldots$

   b) $\cos(180° - \theta) = \ldots\ldots$

   c) $\tan(180° - \theta) = \ldots\ldots$

   d) $\sin(180° + \theta) = \ldots\ldots$

   e) $\cos(180° + \theta) = \ldots\ldots$

   f) $\tan(180° + \theta) = \ldots\ldots$

Worked example 7: Reduction formulae for function values of 180° $\pm\,\theta$

QUESTION

Write the following as a single trigonometric ratio:

$$\frac{\sin 163^\circ}{\cos 197^\circ} + \tan 17^\circ + \cos(180^\circ - \theta) \times \tan(180^\circ + \theta)$$

SOLUTION

Step 1: Use reduction formulae to write the trigonometric function values in terms of acute angles and θ

$$= \frac{\sin(180^\circ - 17^\circ)}{\cos(180^\circ + 17^\circ)} + \tan 17^\circ + (-\cos\theta) \times \tan\theta$$

Step 2: Simplify

$$\begin{aligned}
&= \frac{\sin 17^\circ}{-\cos 17^\circ} + \tan 17^\circ - \cos\theta \times \frac{\sin\theta}{\cos\theta} \\
&= -\tan 17^\circ + \tan 17^\circ - \sin\theta \\
&= -\sin\theta
\end{aligned}$$

Exercise 6 – 3: Reduction formulae for function values of 180° $\pm\,\theta$

1. Determine the value of the following expressions without using a calculator:

   a) $\tan 150^\circ \sin 30^\circ - \cos 210^\circ$

   b) $(1 + \cos 120^\circ)(1 - \sin^2 240^\circ)$

   c) $\cos^2 140^\circ + \sin^2 220^\circ$

2. Write the following in terms of a single trigonometric ratio:

   a) $\tan(180^\circ - \theta) \times \sin(180^\circ + \theta)$

   b) $\dfrac{\tan(180^\circ + \theta)\cos(180^\circ - \theta)}{\sin(180^\circ - \theta)}$

3. If $t = \tan 40°$, express the following in terms of t:

   a) $\tan 140° + 3\tan 220°$

   b) $\dfrac{\cos 220°}{\sin 140°}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

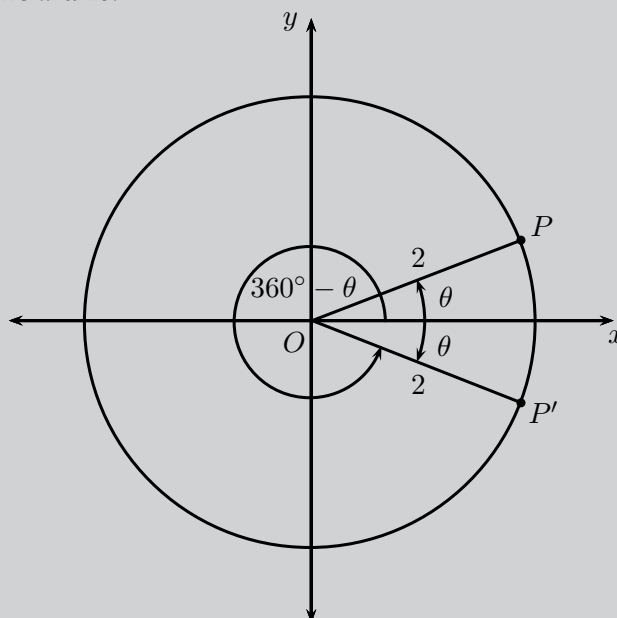
1a. 22YM 1b. 22YN 1c. 22YP 2a. 22YQ 2b. 22YR 3. 22YS

Investigation: Reduction formulae for function values of (360° $\pm\,\theta$) and $(-\theta)$

1. **Function values of (360° $-\,\theta$) and $(-\theta)$**

   In the Cartesian plane we measure angles from the positive x-axis to the terminal arm, which means that an anti-clockwise rotation gives a positive angle. We can therefore measure negative angles by rotating in a clockwise direction.

   For an acute angle θ, we know that $-\theta$ will lie in the fourth quadrant.

   In the figure $P(\sqrt{3}; 1)$ and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis.

   a) If points P and P' are symmetrical about the x-axis ($y = 0$), determine the coordinates of P'.

   b) Use the coordinates of P' to determine $\sin(360° - \theta)$, $\cos(360° - \theta)$ and $\tan(360° - \theta)$.

   c) Use the coordinates of P' to determine $\sin(-\theta)$, $\cos(-\theta)$ and $\tan(-\theta)$.

   d) From your results determine a relationship between the function values of $(360° - \theta)$ and $-\theta$.

e) Complete the following reduction formulae:

i. $\sin(360° - \theta) = \ldots\ldots$

ii. $\cos(360° - \theta) = \ldots\ldots$

iii. $\tan(360° - \theta) = \ldots\ldots$

iv. $\sin(-\theta) = \ldots\ldots$

v. $\cos(-\theta) = \ldots\ldots$

vi. $\tan(-\theta) = \ldots\ldots$

2. **Function values of 360°** $+ \theta$

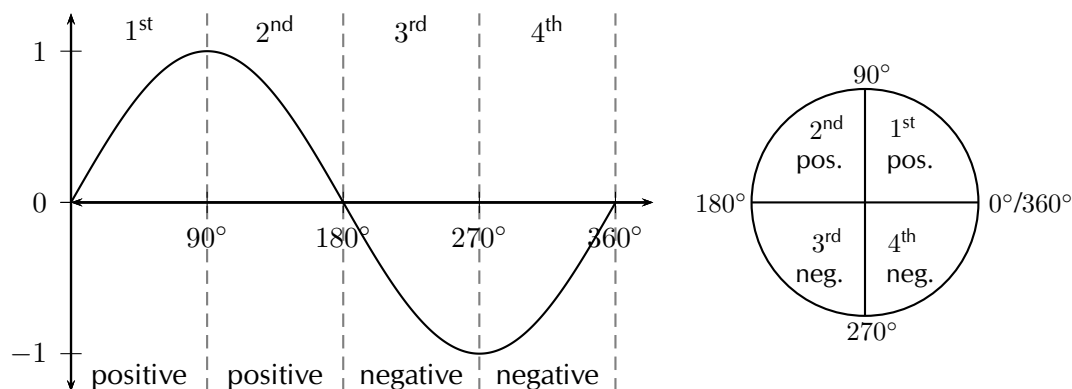
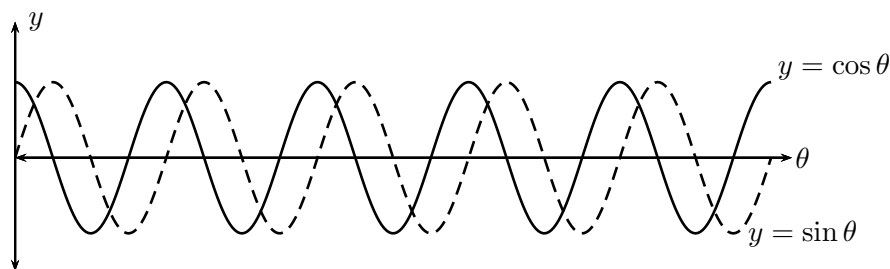
We can also have an angle that is larger than 360°. The angle completes a revolution of 360° and then continues to give an angle of θ.

Complete the following reduction formulae:

a) $\sin(360° + \theta) = \ldots\ldots$

b) $\cos(360° + \theta) = \ldots\ldots$

c) $\tan(360° + \theta) = \ldots\ldots$

From working with functions, we know that the graph of $y = \sin\theta$ has a period of 360°. Therefore, one complete wave of a sine graph is the same as one complete revolution for $\sin\theta$ in the Cartesian plane.

We can also have multiple revolutions. The periodicity of the trigonometric graphs shows this clearly. A complete sine or cosine curve is completed in 360°.

If k is any integer, then

$$\sin(k\,.\,360° + \theta) = \sin\theta$$
$$\cos(k\,.\,360° + \theta) = \cos\theta$$
$$\tan(k\,.\,360° + \theta) = \tan\theta$$

Worked example 8: Reduction formulae for function values of 360° $\pm\,\theta$

QUESTION

If $f = \tan 67°$, express the following in terms of f

$$\frac{\sin 293°}{\cos 427°} + \tan(-67°) + \tan 1147°$$

SOLUTION

Step 1: Using reduction formula

$$= \frac{\sin(360° - 67°)}{\cos(360° + 67°)} - \tan(67°) + \tan\left(3(360°) + 67°\right)$$

$$= \frac{-\sin 67°}{\cos 67°} - \tan 67° + \tan 67°$$

$$= -\tan 67°$$

$$= -f$$

Worked example 9: Using reduction formula

QUESTION

Evaluate without using a calculator:

$$\tan^2 210° - (1 + \cos 120°)\sin^2 405°$$

SOLUTION

Step 1: Simplify the expression using reduction formulae and special angles

$$= \tan^2(180° + 30°) - (1 + \cos(180° - 60°)) \sin^2(360° + 45°)$$

$$= \tan^2 30° - (1 + (-\cos 60°)) \sin^2 45°$$

$$= \left(\frac{1}{\sqrt{3}}\right)^2 - \left(1 - \frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$= \frac{1}{12}$$

Exercise 6 – 4: Using reduction formula

1. Simplify the following:

   a) $\dfrac{\tan(180° - \theta)\sin(360° + \theta)}{\cos(180° + \theta)\tan(360° - \theta)}$

   b) $\cos^2(360° + \theta) + \cos(180° + \theta)\tan(360° - \theta)\sin(360° + \theta)$

   c) $\dfrac{\sin(360° + \alpha)\tan(180° + \alpha)}{\cos(360° - \alpha)\tan^2(360° + \alpha)}$

2. Write the following in terms of $\cos\beta$:
   $\dfrac{\cos(360° - \beta)\cos(-\beta) - 1}{\sin(360° + \beta)\tan(360° - \beta)}$

3. Simplify the following without using a calculator:

   a) $\dfrac{\cos 300° \tan 150°}{\sin 225° \cos(-45°)}$

   b) $3\tan 405° + 2\tan 330° \cos 750°$

   c) $\dfrac{\cos 315° \cos 405° + \sin 45° \sin 135°}{\sin 750°}$

   d) $\tan 150° \cos 390° - 2\sin 510°$

   e) $\dfrac{2\sin 120° + 3\cos 765° - 2\sin 240° - 3\cos 45°}{5\sin 300° + 3\tan 225° - 6\cos 60°}$

4. Given $90° < \alpha < 180°$, use a sketch to help explain why:

   a) $\sin(-\alpha) = -\sin\alpha$

   b) $\cos(-\alpha) = -\cos\alpha$

5. If $t = \sin 43°$, express the following in terms of t:

   a) $\sin 317°$
   b) $\cos^2 403°$
   c) $\tan(-43°)$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22YT 1b. 22YV 1c. 22YW 2. 22YX 3a. 22YY 3b. 22YZ
3c. 22Z2 3d. 22Z3 3e. 22Z4 4. 22Z5 5. 22Z6

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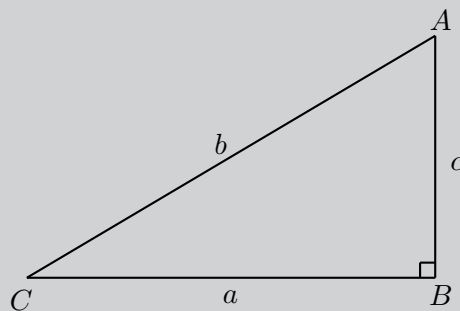
Investigation: Reduction formulae for function values of 90° $\pm \theta$

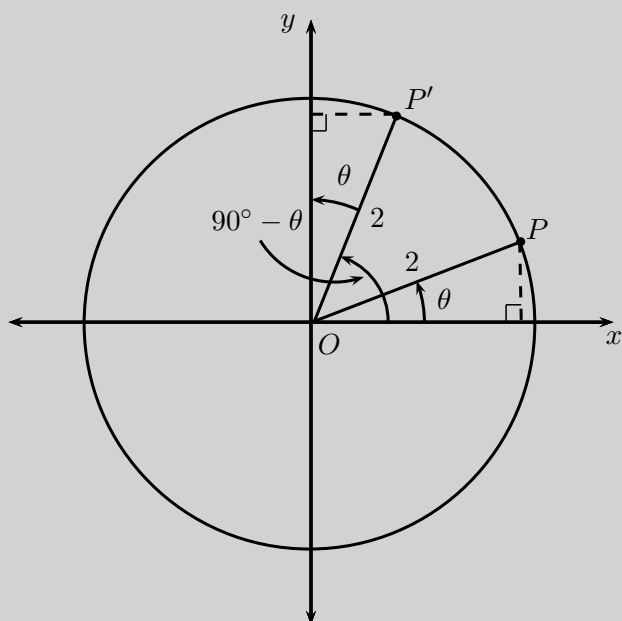
In any right-angled triangle, the two acute angles are complements of each other, $\hat{A} + \hat{C} = 90°$

Complete the following:

In $\triangle ABC$

$$\sin \hat{C} = \frac{c}{b} = \cos \ldots$$

$$\cos \hat{C} = \frac{a}{b} = \sin \ldots$$

Complementary angles are positive acute angles that add up to 90°. For example 20° and 70° are complementary angles.

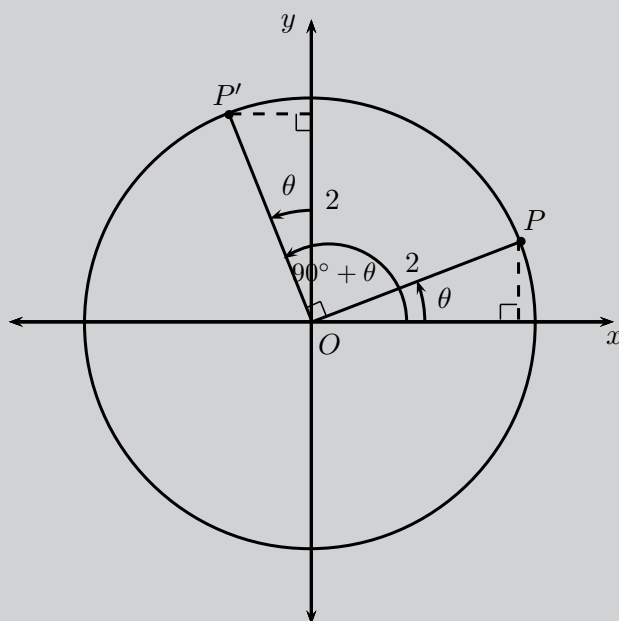
In the figure $P(\sqrt{3};1)$ and P' lie on a circle with radius 2. OP makes an angle of $\theta = 30°$ with the x-axis.

1. **Function values of 90°** $- \theta$

   a) If points P and P' are symmetrical about the line $y = x$, determine the coordinates of P'.

   b) Use the coordinates for P' to determine $\sin(90° - \theta)$ and $\cos(90° - \theta)$.

   c) From your results determine a relationship between the function values of $(90° - \theta)$ and θ.

2. **Function values of 90°** $+ \theta$

   In the figure $P(\sqrt{3};1)$ and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis.

a) If point P is rotated through 90° to get point P', determine the coordinates of P'.

b) Use the coordinates for P' to determine $\sin(90° + \theta)$ and $\cos(90° + \theta)$.

c) From your results determine a relationship between the function values of $(90° + \theta)$ and θ.

3. Complete the following reduction formulae:

a) $\sin(90° - \theta) = \ldots\ldots$

b) $\cos(90° - \theta) = \ldots\ldots$

c) $\sin(90° + \theta) = \ldots\ldots$

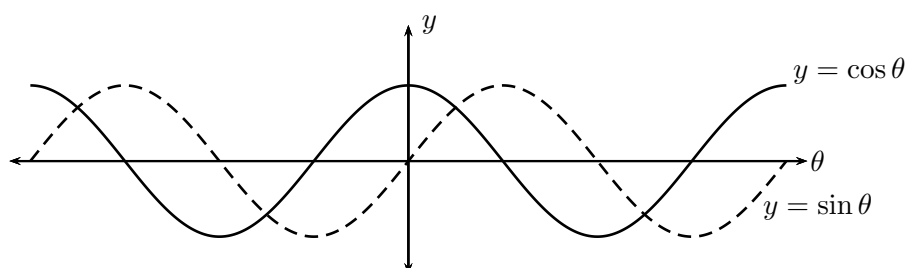
d) $\cos(90° + \theta) = \ldots\ldots$

Sine and cosine are known as **co-functions**. Two functions are called co-functions if $f(A) = g(B)$ whenever $A + B = 90°$ (that is, A and B are complementary angles).

The function value of an angle is equal to the co-function of its complement.

Thus for sine and cosine we have

$$\sin(90° - \theta) = \cos\theta$$
$$\cos(90° - \theta) = \sin\theta$$

The sine and cosine graphs illustrate this clearly: the two graphs are identical except that they have a 90° phase difference.

Worked example 10: Using the co-function rule

QUESTION

Write each of the following in terms of $\sin 40°$:

1. $\cos 50°$
2. $\sin 320°$
3. $\cos 230°$
4. $\cos 130°$

SOLUTION

1. $\cos 50° = \sin(90° - 50°) = \sin 40°$
2. $\sin 320° = \sin(360° - 40°) = -\sin 40°$
3. $\cos 230° = \cos(180° + 50°) = -\cos 50° = -\cos(90° - 40°) = -\sin 40°$
4. $\cos 130° = \cos(90° + 40°) = -\sin 40°$

Function values of $\theta - 90°$

We can write $\sin(\theta - 90°)$ as

$$\begin{aligned} \sin(\theta - 90°) &= \sin\left[-(90° - \theta)\right] \\ &= -\sin(90° - \theta) \\ &= -\cos\theta \end{aligned}$$

similarly, we can show that $\cos(\theta - 90°) = \sin\theta$

Therefore, $\sin(\theta - 90°) = -\cos\theta$ and $\cos(\theta - 90°) = \sin\theta$.

Worked example 11: Co-functions

QUESTION

Express the following in terms of t if $t = \sin\theta$:

$$\frac{\cos(\theta - 90°)\cos(720° + \theta)\tan(\theta - 360°)}{\sin^2(\theta + 360°)\cos(\theta + 90°)}$$

SOLUTION

Step 1: Simplify the expression using reduction formulae and co-functions

Use the CAST diagram to check in which quadrants the trigonometric ratios are positive and negative.

$$\frac{\cos(\theta - 90°)\cos(720° + \theta)\tan(\theta - 360°)}{\sin^2(\theta + 360°)\cos(\theta + 90°)}$$

$$= \frac{\cos[-(90° - \theta)]\cos[2(360°) + \theta]\tan[-(360° - \theta)]}{\sin^2(360° + \theta)\cos(90° + \theta)}$$

$$= \frac{\sin\theta\cos\theta\tan\theta}{\sin^2\theta(-\sin\theta)}$$

$$= -\frac{\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)}{\sin^2\theta}$$

$$= -\frac{1}{\sin\theta}$$

$$= -\frac{1}{t}$$

Exercise 6 – 5: Co-functions

1. Simplify the following:

   a) $\dfrac{\cos(90° + \theta)\sin(\theta + 90°)}{\sin(-\theta)}$

   b) $\dfrac{2\sin(90° - x) + \sin(90° + x)}{\sin(90° - x) + \cos(180° + x)}$

2. Given $\cos 36° = p$, express the following in terms on p:

   a) $\sin 54°$

   b) $\sin 36°$

   c) $\tan 126°$

   d) $\cos 324°$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22Z7 1b. 22Z8 2. 22Z9

Reduction formulae and co-functions:

1. The reduction formulae hold for any angle θ. For convenience, we assume θ is an acute angle ($0° < \theta < 90°$).
2. When determining function values of ($180° \pm \theta$), ($360° \pm \theta$) and ($-\theta$) the function does not change.
3. When determining function values of ($90° \pm \theta$) and ($\theta \pm 90°$) the function changes to its co-function.

second quadrant $(180° - \theta)$ **or** $(90° + \theta)$	**first quadrant** (θ) **or** $(90° - \theta)$
$\sin(180° - \theta) = +\sin\theta$	all trig functions are positive
$\cos(180° - \theta) = -\cos\theta$	$\sin(360° + \theta) = \sin\theta$
$\tan(180° - \theta) = -\tan\theta$	$\cos(360° + \theta) = \cos\theta$
$\sin(90° + \theta) = +\cos\theta$	$\tan(360° + \theta) = \tan\theta$
$\cos(90° + \theta) = -\sin\theta$	$\sin(90° - \theta) = \cos\theta$
	$\cos(90° - \theta) = \sin\theta$
third quadrant $(180° + \theta)$	**fourth quadrant** $(360° - \theta)$
$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = +\cos\theta$
$\tan(180° + \theta) = +\tan\theta$	$\tan(360° - \theta) = -\tan\theta$

Exercise 6 – 6: Reduction formulae

1. Write A and B as a single trigonometric ratio:
   a) $A = \sin(360° - \theta)\cos(180° - \theta)\tan(360° + \theta)$
   b) $B = \dfrac{\cos(360° + \theta)\cos(-\theta)\sin(-\theta)}{\cos(90° + \theta)}$
   c) Hence, determine:
      i. $A + B = \ldots$
      ii. $\frac{A}{B} = \ldots$
2. Write the following as a function of an acute angle:
   a) $\sin 163°$
   b) $\cos 327°$
   c) $\tan 248°$
   d) $\cos(-213°)$
3. Determine the value of the following, without using a calculator:
   a) $\dfrac{\sin(-30°)}{\tan(150°)} + \cos 330°$
   b) $\tan 300° \cos 120°$
   c) $(1 - \cos 30°)(1 - \cos 210°)$
   d) $\cos 780° - (\sin 315°)(\cos 405°)$
4. Prove that the following identity is true and state any restrictions:
   $$\frac{\sin(180° + \alpha)\tan(360° + \alpha)\cos\alpha}{\cos(90° - \alpha)} = \sin\alpha$$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1\. 22ZB 2a. 22ZC 2b. 22ZD 2c. 22ZF 2d. 22ZG 3a. 22ZH
3b. 22ZJ 3c. 22ZK 3d. 22ZM 4. 22ZN

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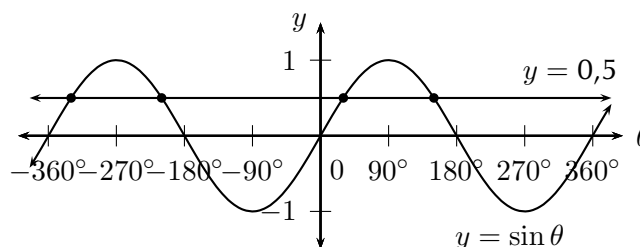
6.4 Trigonometric equations EMBHM

Solving trigonometric equations requires that we find the value of the angles that satisfy the equation. If a specific interval for the solution is given, then we need only find the value of the angles within the given interval that satisfy the equation. If no interval is given, then we need to find the general solution. The periodic nature of trigonometric functions means that there are many values that satisfy a given equation, as shown in the diagram below.

Worked example 12: Solving trigonometric equations

QUESTION

Solve for θ (correct to one decimal place), given $\tan\theta = 5$ and $\theta \in [0°; 360°]$.

SOLUTION

Step 1: Use a calculator to solve for θ

$$\begin{aligned} \tan\theta &= 5 \\ \therefore \theta &= \tan^{-1} 5 \\ &= 78,7° \end{aligned}$$

This value of θ is an acute angle which lies in the first quadrant and is called the reference angle.

Step 2: Use the CAST diagram to determine in which quadrants $\tan\theta$ is positive

The CAST diagram indicates that $\tan\theta$ is positive in the first and third quadrants, therefore we must determine the value of θ such that $180° < \theta < 270°$.

Using reduction formulae, we know that $\tan(180° + \theta) = \tan\theta$

$$\begin{aligned} \theta &= 180° + 78,7° \\ \therefore \theta &= 258,7° \end{aligned}$$

Step 3: Use a calculator to check that the solution satisfies the original equation

Step 4: Write the final answer

$\theta = 78,7°$ or $\theta = 258,7°$.

Worked example 13: Solving trigonometric equations

QUESTION

Solve for α (correct to one decimal place), given $\cos\alpha = -0{,}7$ and $\theta \in [0°; 360°]$.

SOLUTION

Step 1: Use a calculator to find the reference angle

To determine the reference angle, we do not include the negative sign. The reference angle must be an acute angle in the first quadrant, where all the trigonometric functions are positive.

$$\begin{aligned} \text{ref}\,\angle &= \cos^{-1} 0{,}7 \\ &= 45{,}6° \end{aligned}$$

Step 2: Use the CAST diagram to determine in which quadrants $\cos\alpha$ is negative

The CAST diagram indicates that $\cos\alpha$ is negative in the second and third quadrants, therefore we must determine the value of α such that $90° < \alpha < 270°$.

Using reduction formulae, we know that $\cos(180° - \alpha) = -\cos\alpha$ and $\cos(180° + \alpha) = -\cos\alpha$

In the second quadrant:

$$\begin{aligned} \alpha &= 180° - 45{,}6° \\ &= 134{,}4° \end{aligned}$$

In the third quadrant:

$$\begin{aligned} \alpha &= 180° + 45{,}6° \\ &= 225{,}6° \end{aligned}$$

Note: the reference angle $(45{,}6°)$ does not form part of the solution.

Step 3: Use a calculator to check that the solution satisfies the original equation

Step 4: Write the final answer

$\alpha = 134{,}4°$ or $\alpha = 225{,}6°$.

Worked example 14: Solving trigonometric equations

QUESTION

Solve for β (correct to one decimal place), given $\sin\beta = -0{,}5$ and $\beta \in [-360°; 360°]$.

SOLUTION

Step 1: Use a calculator to find the reference angle

To determine the reference angle, we use a positive value.

$$\begin{aligned} \text{ref } \angle &= \sin^{-1} 0{,}5 \\ &= 30° \end{aligned}$$

Step 2: Use the CAST diagram to determine in which quadrants $\sin\beta$ is negative

The CAST diagram indicates that $\sin\beta$ is negative in the third and fourth quadrants. We also need to find the values of β such that $-360° \leq \beta \leq 360°$.

Using reduction formulae, we know that $\sin(180° + \beta) = -\sin\beta$ and $\sin(360° - \beta) = -\sin\beta$

In the third quadrant:

$$\begin{aligned} \beta &= 180° + 30° \\ &= 210° \\ \text{or } \beta &= -180° + 30° \\ &= -150° \end{aligned}$$

In the fourth quadrant:

$$\begin{aligned} \beta &= 360° - 30° \\ &= 330° \\ \text{or } \beta &= 0° - 30° \\ &= -30° \end{aligned}$$

Notice: the reference angle $(30°)$ does not form part of the solution.

Step 3: Use a calculator to check that the solution satisfies the original equation

Step 4: Write the final answer

$\beta = -150°$, $-30°$, $210°$ or $330°$.

Exercise 6 – 7: Solving trigonometric equations

1. Determine the values of α for $\alpha \in [0°; 360°]$ if:

   a) $4\cos\alpha = 2$

   b) $\sin\alpha + 3{,}65 = 3$

   c) $\tan\alpha = 5\frac{1}{4}$

   d) $\cos\alpha + 0{,}939 = 0$

   e) $5\sin\alpha = 3$

   f) $\frac{1}{2}\tan\alpha = -1{,}4$

2. Determine the values of θ for $\theta \in [-360°; 360°]$ if:

   a) $\sin\theta = 0{,}6$

   b) $\cos\theta + \frac{3}{4} = 0$

   c) $3\tan\theta = 20$

   d) $\sin\theta = \cos 180°$

   e) $2\cos\theta = \frac{4}{5}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 22ZP 1b. 22ZQ 1c. 22ZR 1d. 22ZS 1e. 22ZT 1f. 22ZV
2a. 22ZW 2b. 22ZX 2c. 22ZY 2d. 22ZZ 2e. 2322

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The general solution EMBHN

In the previous worked example, the solution was restricted to a certain interval. However, the periodicity of the trigonometric functions means that there are an infinite number of positive and negative angles that satisfy an equation. If we do not restrict the solution, then we need to determine the general solution to the equation. We know that the sine and cosine functions have a period of 360° and the tangent function has a period of 180°.

Method for finding the general solution:

1. Determine the reference angle (use a positive value).
2. Use the CAST diagram to determine where the function is positive or negative (depending on the given equation).
3. Find the angles in the interval $[0°; 360°]$ that satisfy the equation and add multiples of the period to each answer.
4. Check answers using a calculator.

Worked example 15: Finding the general solution

QUESTION

Determine the general solution for $\sin\theta = 0{,}3$ (correct to one decimal place).

SOLUTION

Step 1: Use a calculator to find the reference angle

$$\begin{aligned} \sin\theta &= 0{,}3 \\ \therefore \text{ref } \angle &= \sin^{-1} 0{,}3 \\ &= 17{,}5° \end{aligned}$$

Step 2: Use CAST diagram to determine in which quadrants $\sin\theta$ is positive

The CAST diagram indicates that $\sin\theta$ is positive in the first and second quadrants.

Using reduction formulae, we know that $\sin(180° - \theta) = \sin\theta$.

In the first quadrant:

$$\begin{aligned} \theta &= 17{,}5° \\ \therefore \theta &= 17{,}5° + k\ .\ 360° \end{aligned}$$

In the second quadrant:

$$\begin{aligned} \theta &= 180° - 17{,}5° \\ \therefore \theta &= 162{,}5° + k\ .\ 360° \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Check that the solution satisfies the original equation

We can select random values of k to check that the answers satisfy the original equation.

Let $k = 4$:

$$\begin{aligned} \theta &= 17{,}5° + 4(360)° \\ \therefore \theta &= 1457{,}5° \\ \text{And } \sin 1457{,}5° &= 0{,}3007\ldots \end{aligned}$$

This solution is correct.

Similarly, if we let $k = -2$:

$$\begin{aligned} \theta &= 162{,}5° - 2(360)° \\ \therefore \theta &= -557{,}5° \\ \text{And } \sin(-557{,}5°) &= 0{,}3007\ldots \end{aligned}$$

This solution is also correct.

Step 4: Write the final answer

$\theta = 17{,}5° + k\ .\ 360°$ or $\theta = 162{,}5° + k\ .\ 360°$.

Worked example 16: Finding the general solution

QUESTION

Determine the general solution for $\cos 2\theta = -0{,}6427$ (give answers correct to one decimal place).

SOLUTION

Step 1: Use a calculator to find the reference angle

$$\begin{aligned} \text{ref } \angle &= \sin^{-1} 0{,}6427 \\ &= 50{,}0^\circ \end{aligned}$$

Step 2: Use CAST diagram to determine in which quadrants $\cos\theta$ is negative

The CAST diagram shows that $\cos\theta$ is negative in the second and third quadrants.

Therefore we use the reduction formulae $\cos(180^\circ - \theta) = -\cos\theta$ and $\cos(180^\circ + \theta) = -\cos\theta$.

In the second quadrant:

$$\begin{aligned} 2\theta &= 180^\circ - 50^\circ + k\ .\ 360^\circ \\ &= 130^\circ + k\ .\ 360^\circ \\ \therefore \theta &= 65^\circ + k\ .\ 180^\circ \end{aligned}$$

In the third quadrant:

$$\begin{aligned} 2\theta &= 180^\circ + 50^\circ + k\ .\ 360^\circ \\ &= 230^\circ + k\ .\ 360^\circ \\ \therefore \theta &= 115^\circ + k\ .\ 180^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Remember: also divide the period (360°) by the coefficient of θ.

Step 3: Check that the solution satisfies the original equation

We can select random values of k to check that the answers satisfy the original equation.

Let $k = 2$:

$$\begin{aligned} \theta &= 65^\circ + 2(180^\circ) \\ \therefore \theta &= 425^\circ \\ \text{And } \cos 2(425)^\circ &= -0{,}6427\ldots \end{aligned}$$

This solution is correct.

Similarly, if we let $k = -5$:

$$\theta = 115° - 5(180°)$$
$$\therefore \theta = -785°$$
$$\text{And } \cos 2(-785°) = -0{,}6427\ldots$$

This solution is also correct.

Step 4: Write the final answer

$\theta = 65° + k \;.\; 180°$ or $\theta = 115° + k \;.\; 180°$.

Worked example 17: Finding the general solution

QUESTION

Determine the general solution for $\tan(2\alpha - 10°) = 2{,}5$ such that $-180° \leq \alpha \leq 180°$ (give answers correct to one decimal place).

SOLUTION

Step 1: Make a substitution

To solve this equation, it can be useful to make a substitution: let $x = 2\alpha - 10°$.

$$\tan(x) = 2{,}5$$

Step 2: Use a calculator to find the reference angle

$$\tan x = 2{,}5$$
$$\therefore \text{ref } \angle = \tan^{-1} 2{,}5$$
$$= 68{,}2°$$

Step 3: Use CAST diagram to determine in which quadrants the tangent function is positive

We see that $\tan x$ is positive in the first and third quadrants, so we use the reduction formula $\tan(180° + x) = \tan x$. It is also important to remember that the period of the tangent function is 180°.

In the first quadrant:

$$\begin{aligned} x &= 68{,}2^\circ + k\ .\ 180^\circ \\ \text{Substitute } x &= 2\alpha - 10^\circ \\ 2\alpha - 10^\circ &= 68{,}2^\circ + k\ .\ 180^\circ \\ 2\alpha &= 78{,}2^\circ + k\ .\ 180^\circ \\ \therefore \alpha &= 39{,}1^\circ + k\ .\ 90^\circ \end{aligned}$$

In the third quadrant:

$$\begin{aligned} x &= 180^\circ + 68{,}2^\circ + k\ .\ 180^\circ \\ &= 248{,}2^\circ + k\ .\ 180^\circ \\ \text{Substitute } x &= 2\alpha - 10^\circ \\ 2\alpha - 10^\circ &= 248{,}2^\circ + k\ .\ 180^\circ \\ 2\alpha &= 258{,}2^\circ + k\ .\ 180^\circ \\ \therefore \alpha &= 129{,}1^\circ + k\ .\ 90^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Remember: to divide the period (180°) by the coefficient of α.

Step 4: Find the answers within the given interval

Substitute suitable values of k to determine the values of α that lie within the interval $(-180^\circ \leq \alpha \leq 180^\circ)$.

	I: $\alpha = 39{,}1^\circ + k\ .\ 90^\circ$		III: $\alpha = 129{,}1^\circ + k\ .\ 90^\circ$	
$k = 0$	$39{,}1^\circ$		$129{,}1^\circ$	
$k = 1$	$129{,}1^\circ$		$219{,}1^\circ$	(outside)
$k = 2$	$219{,}1^\circ$	(outside)		
$k = -1$	$-50{,}9^\circ$		$39{,}1^\circ$	
$k = -2$	$-140{,}9^\circ$		$-50{,}9^\circ$	
$k = -3$	$-230{,}9^\circ$	(outside)	$-140{,}9^\circ$	
$k = -4$			$-230{,}9^\circ$	(outside)

Notice how some of the values repeat. This is because of the periodic nature of the tangent function. Therefore we need only determine the solution:

$$\alpha = 39{,}1^\circ + k\ .\ 90^\circ$$

for $k \in \mathbb{Z}$.

Step 5: Write the final answer

$\alpha = -140{,}9^\circ; -50{,}9^\circ; 39{,}1^\circ$ or $129{,}1^\circ$.

Worked example 18: Finding the general solution using co-functions

QUESTION

Determine the general solution for $\sin(\theta - 20°) = \cos 2\theta$.

SOLUTION

Step 1: Use co-functions to simplify the equation

$$\begin{aligned}
\sin(\theta - 20°) &= \cos 2\theta \\
&= \sin(90° - 2\theta) \\
\therefore \theta - 20° &= 90° - 2\theta + k\ .\ 360°, \quad k \in \mathbb{Z} \\
3\theta &= 110° + k\ .\ 360° \\
\therefore \theta &= 36{,}7° + k\ .\ 120°
\end{aligned}$$

Step 2: Use the CAST diagram to determine the correct quadrants

Since the original equation equates a sine and cosine function, we need to work in the quadrant where both functions are positive or in the quadrant where both functions are negative so that the equation holds true. We therefore determine the solution using the first and third quadrants.

In the first quadrant: $\theta = 36{,}7° + k\ .\ 120°$.

In the third quadrant:

$$\begin{aligned}
3\theta &= 180° + 110° + k\ .\ 360° \\
&= 290° + k\ .\ 360° \\
\therefore \theta &= 96{,}6° + k\ .\ 120°
\end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Check that the solution satisfies the original equation

Step 4: Write the final answer

$\theta = 36{,}7° + k\ .\ 120°$ or $\theta = 96{,}6° + k\ .\ 120°$

Exercise 6 – 8: General solution

1.
   - Find the general solution for each equation.
   - Hence, find all the solutions in the interval $[-180°; 180°]$.

a) $\cos(\theta + 25°) = 0{,}231$

b) $\sin 2\alpha = -0{,}327$

c) $2 \tan \beta = -2{,}68$

d) $\cos \alpha = 1$

e) $4 \sin \theta = 0$

f) $\cos \theta = -1$

g) $\tan \frac{\theta}{2} = 0{,}9$

h) $4 \cos \theta + 3 = 1$

i) $\sin 2\theta = -\frac{\sqrt{3}}{2}$

2. Find the general solution for each equation.

a) $\cos(\theta + 20°) = 0$

b) $\sin 3\alpha = -1$

c) $\tan 4\beta = 0{,}866$

d) $\cos(\alpha - 25°) = 0{,}707$

e) $2 \sin \frac{3\theta}{2} = -1$

f) $5 \tan(\beta + 15°) = \frac{5}{\sqrt{3}}$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 2323 1b. 2324 1c. 2325 1d. 2326 1e. 2327 1f. 2328
1g. 2329 1h. 232B 1i. 232C 2a. 232D 2b. 232F 2c. 232G
2d. 232H 2e. 232J 2f. 232K

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Solving quadratic trigonometric equations

We can use our knowledge of algebraic equations to solve quadratic trigonometric equations.

Worked example 19: Quadratic trigonometric equations

QUESTION

Find the general solution of $4 \sin^2 \theta = 3$.

SOLUTION

Step 1: Simplify the equation and determine the reference angle

$$\begin{aligned} 4\sin^2\theta &= 3 \\ \sin^2\theta &= \frac{3}{4} \\ \therefore \sin\theta &= \pm\sqrt{\frac{3}{4}} \\ &= \pm\frac{\sqrt{3}}{2} \\ \therefore \text{ref } \angle &= 60° \end{aligned}$$

Step 2: Determine in which quadrants the sine function is positive and negative

The CAST diagram shows that $\sin\theta$ is positive in the first and second quadrants and negative in the third and fourth quadrants.

Positive in the first and second quadrants:

$$\begin{aligned} \theta &= 60° + k \,.\, 360° \\ \text{or } \theta &= 180° - 60° + k \,.\, 360° \\ &= 120° + k \,.\, 360° \end{aligned}$$

Negative in the third and fourth quadrants:

$$\begin{aligned} \theta &= 180° + 60° + k \,.\, 360° \\ &= 240° + k \,.\, 360° \\ \text{or } \theta &= 360° - 60° + k \,.\, 360° \\ &= 300° + k \,.\, 360° \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Check that the solution satisfies the original equation

Step 4: Write the final answer

$\theta = 60° + k \,.\, 360°$ or $120° + k \,.\, 360°$ or $240° + k \,.\, 360°$ or $300° + k \,.\, 360°$

Worked example 20: Quadratic trigonometric equations

QUESTION

Find θ if $2\cos^2\theta - \cos\theta - 1 = 0$ for $\theta \in [-180°; 180°]$.

SOLUTION

Step 1: Factorise the equation

$$\begin{aligned} 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ \therefore 2\cos\theta + 1 &= 0 \text{ or } \cos\theta - 1 = 0 \end{aligned}$$

Step 2: Simplify the equations and solve for θ

$$\begin{aligned} 2\cos\theta + 1 &= 0 \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \therefore \text{ref } \angle &= 60^\circ \\ \text{II quadrant: } \theta &= 180^\circ - 60^\circ + k \, . \, 360^\circ \\ &= 120^\circ + k \, . \, 360^\circ \\ \text{III quadrant: } \theta &= 180^\circ + 60^\circ + k \, . \, 360^\circ \\ &= 240^\circ + k \, . \, 360^\circ \end{aligned}$$

or

$$\begin{aligned} \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \\ \therefore \text{ref } \angle &= 0^\circ \\ \text{II and IV quadrants: } \theta &= k \, . \, 360^\circ \end{aligned}$$

where $k \in \mathbb{Z}$.

Step 3: Substitute suitable values of k

Determine the values of θ that lie within the the given interval $\theta \in [-180^\circ; 180^\circ]$ by substituting suitable values of k.

If $k = -1$,

$$\begin{aligned} \theta &= 240^\circ + k \, . \, 360^\circ \\ &= 240^\circ - (360^\circ) \\ &= -120^\circ \end{aligned}$$

If $k = 0$,

$$\begin{aligned} \theta &= 120^\circ + k \, . \, 360^\circ \\ &= 120^\circ + 0(360^\circ) \\ &= 120^\circ \end{aligned}$$

If $k = 1$,

$$\begin{aligned} \theta &= k \, . \, 360^\circ \\ &= 0(360^\circ) \\ &= 0^\circ \end{aligned}$$

Step 4: Alternative method: substitution

We can simplify the given equation by letting $y = \cos\theta$ and then factorising as:

$$\begin{aligned} 2y^2 - y - 1 &= 0 \\ (2y+1)(y-1) &= 0 \\ \therefore y &= -\frac{1}{2} \text{ or } y = 1 \end{aligned}$$

We substitute $y = \cos\theta$ back into these two equations and solve for θ.

Step 5: Write the final answer

$\theta = -120°; 0°; 120°$

Worked example 21: Quadratic trigonometric equations

QUESTION

Find α if $2\sin^2\alpha - \sin\alpha\cos\alpha = 0$ for $\alpha \in [0°; 360°]$.

SOLUTION

Step 1: Factorise the equation by taking out a common factor

$$\begin{aligned} 2\sin^2\alpha - \sin\alpha\cos\alpha &= 0 \\ \sin\alpha(2\sin\alpha - \cos\alpha) &= 0 \\ \therefore \sin\alpha &= 0 \text{ or } 2\sin\alpha - \cos\alpha = 0 \end{aligned}$$

Step 2: Simplify the equations and solve for α

$$\begin{aligned} \sin\alpha &= 0 \\ \therefore \text{ref } \angle &= 0° \\ \therefore \alpha &= 0° + k \,.\, 360° \\ \text{or } \alpha &= 180° + k \,.\, 360° \\ \text{and since } 360° &= 2 \times 180° \\ \text{we therefore have } \alpha &= k \,.\, 180° \end{aligned}$$

or

$$2\sin\alpha - \cos\alpha = 0$$
$$2\sin\alpha = \cos\alpha$$

To simplify further, we divide both sides of the equation by $\cos\alpha$.

$$\frac{2\sin\alpha}{\cos\alpha} = \frac{\cos\alpha}{\cos\alpha} \qquad (\cos\alpha \neq 0)$$
$$2\tan\alpha = 1$$
$$\tan\alpha = \frac{1}{2}$$
$$\therefore \text{ref } \angle = 26{,}6^\circ$$
$$\therefore \alpha = 26{,}6^\circ + k \ . \ 180^\circ$$

where $k \in \mathbb{Z}$.

Step 3: Substitute suitable values of k

Determine the values of α that lie within the the given interval $\alpha \in [0^\circ; 360^\circ]$ by substituting suitable values of k.

If $k = 0$:

$$\alpha = 0^\circ$$
$$\text{or } \alpha = 26{,}6^\circ$$

If $k = 1$:

$$\alpha = 180^\circ$$
$$\text{or } \alpha = 26{,}6^\circ + 180^\circ$$
$$= 206{,}6^\circ$$

If $k = 2$:

$$\alpha = 360^\circ$$

Step 4: Write the final answer

$\alpha = 0^\circ;\ 26{,}6^\circ;\ 180^\circ;\ 206{,}6^\circ;\ 360^\circ$

Exercise 6 – 9: Solving trigonometric equations

1. Find the general solution for each of the following equations:

   a) $\cos 2\theta = 0$
   b) $\sin(\alpha + 10°) = \frac{\sqrt{3}}{2}$
   c) $2\cos\frac{\theta}{2} - \sqrt{3} = 0$
   d) $\frac{1}{2}\tan(\beta - 30°) = -1$
   e) $5\cos\theta = \tan 300°$
   f) $3\sin\alpha = -1{,}5$
   g) $\sin 2\beta = \cos(\beta + 20°)$
   h) $0{,}5\tan\theta + 2{,}5 = 1{,}7$
   i) $\sin(3\alpha - 10°) = \sin(\alpha + 32°)$
   j) $\sin 2\beta = \cos 2\beta$

2. Find θ if $\sin^2\theta + \frac{1}{2}\sin\theta = 0$ for $\theta \in [0°; 360°]$.

3. Determine the general solution for each of the following:

   a) $2\cos^2\theta - 3\cos\theta = 2$
   b) $3\tan^2\theta + 2\tan\theta = 0$
   c) $\cos^2\alpha = 0{,}64$
   d) $\sin(4\beta + 35°) = \cos(10° - \beta)$
   e) $\sin(\alpha + 15°) = 2\cos(\alpha + 15°)$
   f) $\sin^2\theta - 4\cos^2\theta = 0$
   g) $\dfrac{\cos(2\theta + 30°)}{2} + 0{,}38 = 0$

4. Find β if $\frac{1}{3}\tan\beta = \cos 200°$ for $\beta \in [-180°; 180°]$.

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1a. 232M 1b. 232N 1c. 232P 1d. 232Q 1e. 232R 1f. 232S
1g. 232T 1h. 232V 1i. 232W 1j. 232X 2. 232Y 3a. 232Z
3b. 2332 3c. 2333 3d. 2334 3e. 2335 3f. 2336 3g. 2337
4. 2338

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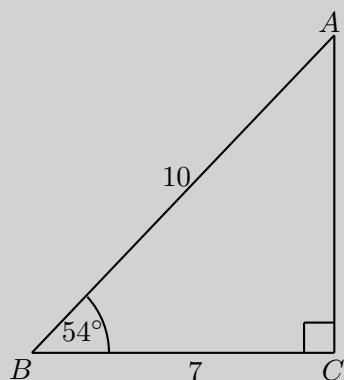
6.5 Area, sine, and cosine rules EMBHP

There are three identities relating to the trigonometric functions that make working with triangles easier:

1. the area rule
2. the sine rule
3. the cosine rule

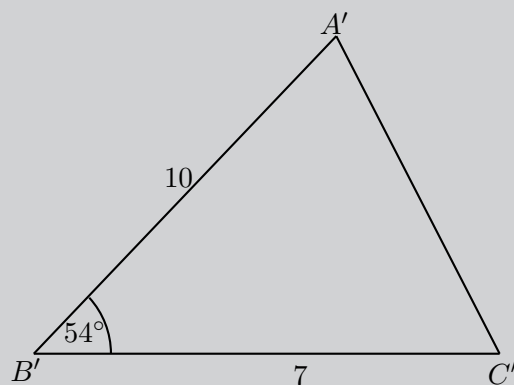
Investigation: The area rule

1. Consider $\triangle ABC$:

Complete the following:

a) Area $\triangle ABC = \frac{1}{2} \times \ldots \times AC$

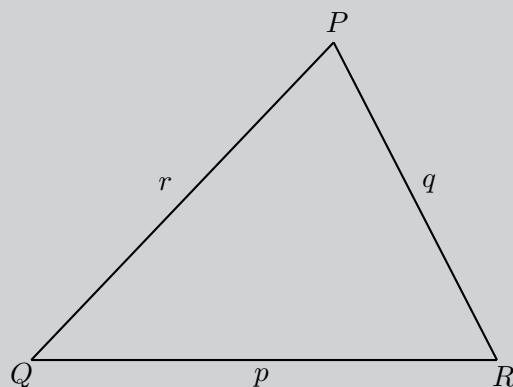
b) $\sin \hat{B} = \ldots$ and $AC = \ldots \times \ldots$

c) Therefore area $\triangle ABC = \ldots \times \ldots \times \ldots \times \ldots$

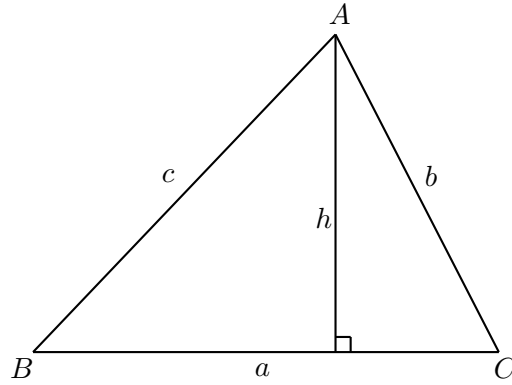
2. Consider $\triangle A'B'C'$:

Complete the following:

a) How is $\triangle A'B'C'$ different from $\triangle ABC$?

b) Calculate area $\triangle A'B'C'$.

3. Use your results to write a general formula for determining the area of $\triangle PQR$:

For any $\triangle ABC$ with $AB = c$, $BC = a$ and $AC = b$, we can construct a perpendicular height (h) from vertex A to the line BC:

In $\triangle ABC$:

$$\sin \hat{B} = \frac{h}{c}$$
$$\therefore h = c \sin \hat{B}$$

And we know that

$$\text{Area } \triangle ABC = \frac{1}{2} \times a \times h$$
$$= \frac{1}{2} \times a \times c \sin \hat{B}$$
$$\therefore \text{Area } \triangle ABC = \frac{1}{2} ac \sin \hat{B}$$

Alternatively, we could write that

$$\sin \hat{C} = \frac{h}{b}$$
$$\therefore h = b \sin \hat{C}$$

And then we would have that

$$\text{Area } \triangle ABC = \frac{1}{2} \times a \times h$$
$$= \frac{1}{2} ab \sin \hat{C}$$

Similarly, by constructing a perpendicular height from vertex B to the line AC, we can also show that area $\triangle ABC = \frac{1}{2} bc \sin \hat{A}$.

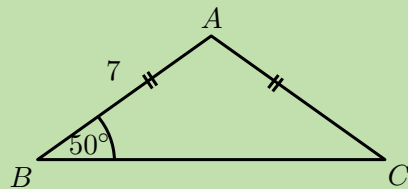
The area rule

In any $\triangle ABC$:

$$\text{Area } \triangle ABC = \frac{1}{2} bc \sin \hat{A}$$
$$= \frac{1}{2} ac \sin \hat{B}$$
$$= \frac{1}{2} ab \sin \hat{C}$$

Worked example 22: The area rule

QUESTION

Find the area of $\triangle ABC$ (correct to two decimal places):

SOLUTION

Step 1: Use the given information to determine unknown angles and sides

$$
\begin{aligned}
AB = AC &= 7 && \text{(given)}\\
\therefore \hat{B} = \hat{C} &= 50° && \text{(}\angle\text{s opp. equal sides)}\\
\text{And } \hat{A} &= 180° - 50° - 50° && \text{(}\angle\text{s sum of } \triangle ABC\text{)}\\
\therefore \hat{A} &= 80°
\end{aligned}
$$

Step 2: Use the area rule to calculate the area of $\triangle ABC$

Notice that we do not know the length of side a and must therefore choose the form of the area rule that does not include this side of the triangle.

In $\triangle ABC$:

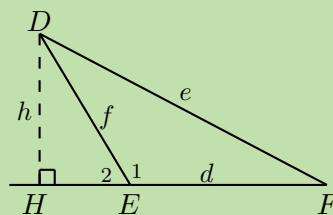
$$
\begin{aligned}
\text{Area} &= \tfrac{1}{2}bc\sin\hat{A}\\
&= \tfrac{1}{2}(7)(7)\sin 80°\\
&= 24{,}13
\end{aligned}
$$

Step 3: Write the final answer

Area of $\triangle ABC = 24{,}13$ square units.

Worked example 23: The area rule

QUESTION

Show that the area of $\triangle DEF = \frac{1}{2}df\sin\hat{E}$.

SOLUTION

Step 1: Construct a perpendicular height h

Draw DH such that $DH \perp EF$ and let $DH = h$, $D\hat{E}F = \hat{E}_1$ and $D\hat{E}H = \hat{E}_2$.

In $\triangle DHE$:

$$\sin\hat{E}_2 = \frac{h}{f}$$

$$h = f\sin(180° - \hat{E}_1) \qquad (\angle\text{s on str. line})$$

$$= f\sin\hat{E}_1$$

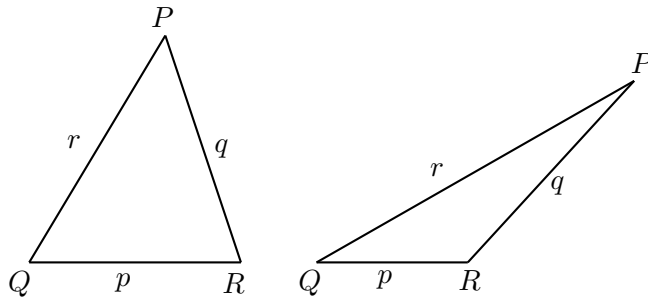
Step 2: Use the area rule to calculate the area of $\triangle DEF$

In $\triangle DEF$:

$$\text{Area } = \frac{1}{2}d \times h$$

$$= \frac{1}{2}df\sin\hat{E}_1$$

The area rule

In any $\triangle PQR$:

$$\text{Area } \triangle PQR = \frac{1}{2}qr\sin\hat{P}$$
$$= \frac{1}{2}pr\sin\hat{Q}$$
$$= \frac{1}{2}pq\sin\hat{R}$$

The area rule states that the area of any triangle is equal to half the product of the lengths of the two sides of the triangle multiplied by the sine of the angle included by the two sides.

Exercise 6 – 10: The area rule

1. Draw a sketch and calculate the area of $\triangle PQR$ given:

   a) $\hat{Q} = 30°$; $r = 10$ and $p = 7$

   b) $\hat{R} = 110°$; $p = 8$ and $q = 9$

2. Find the area of $\triangle XYZ$ given $XZ = 52$ cm, $XY = 29$ cm and $\hat{X} = 58{,}9°$.

3. Determine the area of a parallelogram in which two adjacent sides are 10 cm and 13 cm and the angle between them is 55°.

4. If the area of $\triangle ABC$ is $5000\,\text{m}^2$ with $a = 150\,\text{m}$ and $b = 70\,\text{m}$, what are the two possible sizes of $\hat{C}$?

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1a. 2339 1b. 233B 2. 233C 3. 233D 4. 233F

The sine rule

So far we have only applied the trigonometric ratios to right-angled triangles. We now expand the application of the trigonometric ratios to triangles that do not have a right angle:

Investigation: The sine rule

In $\triangle ABC$, $AC = 15$, $BC = 11$ and $\hat{A} = 48°$. Find $\hat{B}$.

1. Method 1: using the sine ratio

   a) Draw a sketch of $\triangle ABC$.

   b) Construct $CF \perp AB$.

   c) In $\triangle CBF$:

$$\frac{CF}{\ldots} = \sin \hat{B}$$
$$\therefore CF = \ldots \times \sin \hat{B}$$

   d) In $\triangle CAF$:

$$\frac{CF}{15} = \ldots$$
$$\therefore CF = 15 \times \ldots$$

   e) Therefore we have that:

$$CF = 15 \times \ldots$$
$$\text{and } CF = \ldots \times \sin \hat{B}$$
$$\therefore 15 \times \ldots = \ldots \times \sin \hat{B}$$
$$\therefore \sin \hat{B} = \ldots\ldots\ldots$$
$$\therefore \hat{B} = \ldots$$

2. Method 2: using the area rule

a) In $\triangle ABC$:

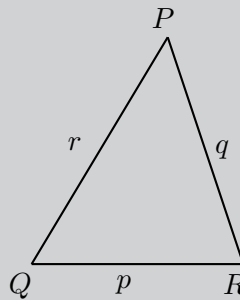
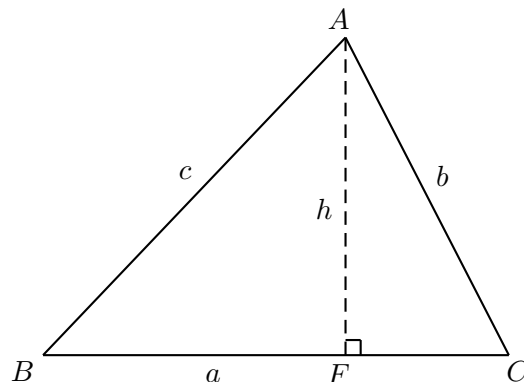
$$\begin{aligned}\text{Area } \triangle ABC &= \frac{1}{2} AB \times AC \times \ldots \\ &= \frac{1}{2} AB \times \ldots \times \ldots\end{aligned}$$

b) And we also know that

$$\text{Area } \triangle ABC = \frac{1}{2} AB \times \ldots \times \sin \hat{B}$$

c) We can equate these two equations and solve for $\hat{B}$:

$$\begin{aligned}\frac{1}{2} AB \times \ldots \times \sin \hat{B} &= \frac{1}{2} AB \times \ldots \times \ldots \\ \therefore \ldots \times \sin \hat{B} &= \ldots \times \ldots \\ \therefore \sin \hat{B} &= \ldots \times \ldots \\ \therefore \hat{B} &= \ldots\end{aligned}$$

3. Use your results to write a general formula for the sine rule given $\triangle PQR$:

For any triangle ABC with $AB = c$, $BC = a$ and $AC = b$, we can construct a perpendicular height (h) at F:

Method 1: using the sine ratio

In $\triangle ABF$:

$$\begin{aligned}\sin \hat{B} &= \frac{h}{c} \\ \therefore h &= c \sin \hat{B}\end{aligned}$$

In $\triangle ACF$:

$$\sin \hat{C} = \frac{h}{b}$$
$$\therefore h = b \sin \hat{C}$$

We can equate the two equations

$$c \sin \hat{B} = b \sin \hat{C}$$
$$\therefore \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$
$$\text{or } \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

Similarly, by constructing a perpendicular height from vertex B to the line AC, we can also show that:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c}$$
$$\text{or } \frac{a}{\sin \hat{A}} = \frac{c}{\sin \hat{C}}$$

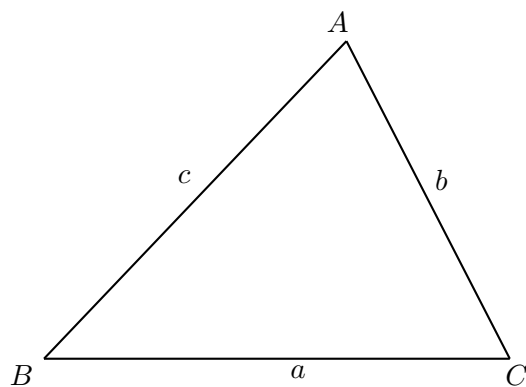
Similarly, by constructing a perpendicular height from vertex B to the line AC, we can also show that area $\triangle ABC = \frac{1}{2} bc \sin \hat{A}$.

Method 2: using the area rule

In $\triangle ABC$:

$$\text{Area } \triangle ABC = \frac{1}{2} ac \sin \hat{B}$$
$$= \frac{1}{2} ab \sin \hat{C}$$
$$\therefore \frac{1}{2} ac \sin \hat{B} = \frac{1}{2} ab \sin \hat{C}$$
$$c \sin \hat{B} = b \sin \hat{C}$$
$$\frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$
$$\text{or } \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

The sine rule

In any $\triangle ABC$:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

See video: 233G at www.everythingmaths.co.za

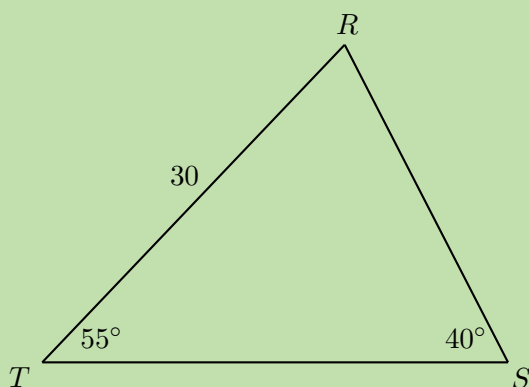
Worked example 24: The sine rule

QUESTION

Given $\triangle TRS$ with $S\hat{T}R = 55°$, $TR = 30$ and $R\hat{S}T = 40°$, determine RS, ST and $T\hat{R}S$.

SOLUTION

Step 1: Draw a sketch

Let $RS = t$, $ST = r$ and $TR = s$.

Step 2: Find $T\hat{R}S$ using angles in a triangle

$$T\hat{R}S + R\hat{S}T + S\hat{T}R = 180° \qquad (\angle\text{s sum of } \triangle TRS)$$

$$\therefore T\hat{R}S = 180° - 40° - 55°$$

$$= 85°$$

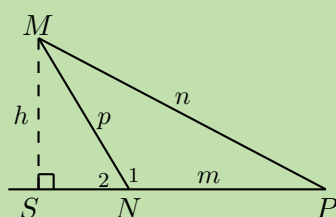
Step 3: Determine t and r using the sine rule

$$\begin{aligned}\frac{t}{\sin \hat{T}} &= \frac{s}{\sin \hat{S}} \\ \frac{t}{\sin 55^\circ} &= \frac{30}{\sin 40^\circ} \\ \therefore t &= \frac{30}{\sin 40^\circ} \times \sin 55^\circ \\ &= 38{,}2\end{aligned}$$

$$\begin{aligned}\frac{r}{\sin \hat{R}} &= \frac{s}{\sin \hat{S}} \\ \frac{r}{\sin 85^\circ} &= \frac{30}{\sin 40^\circ} \\ \therefore r &= \frac{30}{\sin 40^\circ} \times \sin 85^\circ \\ &= 46{,}5\end{aligned}$$

Worked example 25: The sine rule

QUESTION

Prove the sine rule for $\triangle MNP$ with $MS \perp NP$.

SOLUTION

Step 1: Use the sine ratio to express the angles in the triangle in terms of the length of the sides

In $\triangle MSN$:

$$\begin{aligned}\sin \hat{N}_2 &= \tfrac{h}{p} \\ \therefore h &= p \sin \hat{N}_2 \\ \text{and } \hat{N}_2 &= 180^\circ - \hat{N}_1 && \angle\text{s on str. line} \\ \therefore h &= p \sin(180^\circ - \hat{N}_1) \\ &= p \sin \hat{N}_1\end{aligned}$$

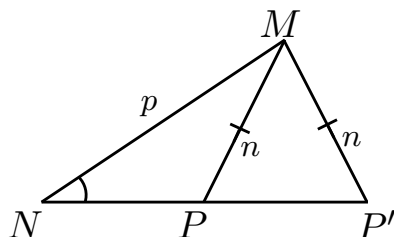
In $\triangle MSP$:

$$\sin \hat{P} = \frac{h}{n}$$
$$\therefore h = n \sin \hat{P}$$

Step 2: Equate the two equations to derive the sine rule

$$p \sin \hat{N}_1 = n \sin \hat{P}$$
$$\therefore \frac{\sin \hat{N}_1}{n} = \frac{\sin \hat{P}}{p}$$
$$\text{or } \frac{n}{\sin \hat{N}_1} = \frac{p}{\sin \hat{P}}$$

The ambiguous case

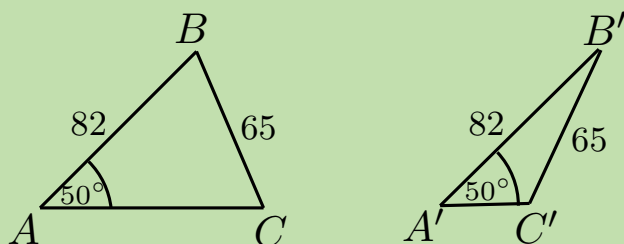
If two sides and an interior angle of a triangle are given, and the side opposite the given angle is the shorter of the two sides, then we can draw two different triangles ($\triangle NMP$ and $\triangle NMP'$), both having the given dimensions. We call this the ambiguous case because there are two ways of interpreting the given information and it is not certain which is the required solution.

Worked example 26: The ambiguous case

QUESTION

In $\triangle ABC$, $AB = 82$, $BC = 65$ and $\hat{A} = 50°$. Draw $\triangle ABC$ and find $\hat{C}$ (correct to one decimal place).

SOLUTION

Step 1: Draw a sketch and identify the ambiguous case

We notice that for the given dimensions of $\triangle ABC$, the side BC opposite $\hat{A}$ is shorter than AB. This means that we can draw two different triangles with the given dimensions.

Step 2: Solve for unknown angle using the sine rule

In $\triangle ABC$:

$$\begin{aligned} \frac{\sin \hat{A}}{BC} &= \frac{\sin \hat{C}}{AB} \\ \frac{\sin 50°}{65} &= \frac{\sin \hat{C}}{82} \\ \therefore \frac{\sin 50°}{65} \times 82 &= \sin \hat{C} \\ \therefore \hat{C} &= 75{,}1° \end{aligned}$$

In $\triangle A'B'C'$:

We know that $\sin(180 - \hat{C}) = \sin \hat{C}$, which means we can also have the solution

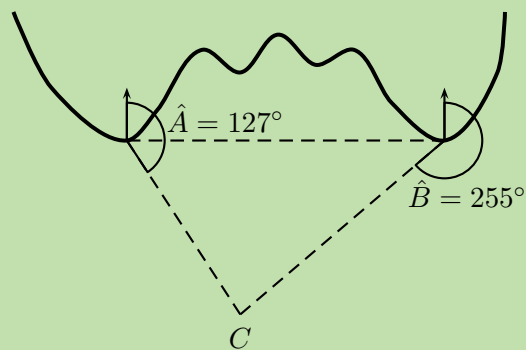
$$\begin{aligned} \hat{C}' &= 180° - 75{,}1° \\ &= 104{,}9° \end{aligned}$$

Both solutions are correct.

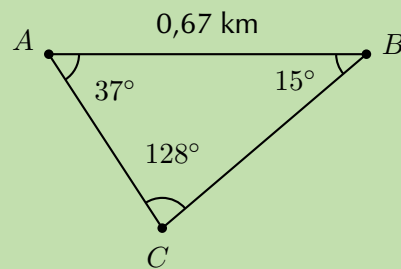
Worked example 27: Lighthouses

QUESTION

There is a coastline with two lighthouses, one on either side of a beach. The two lighthouses are 0,67 km apart and one is exactly due east of the other. The lighthouses tell how close a boat is by taking bearings to the boat (a bearing is an angle measured clockwise from north). These bearings are shown on the diagram below.

Calculate how far the boat is from each lighthouse.

SOLUTION

We see that the two lighthouses and the boat form a triangle. Since we know the distance between the lighthouses and we have two angles we can use trigonometry

to find the remaining two sides of the triangle, the distance of the boat from the two lighthouses.

We need to determine the lengths of the two sides AC and BC. We can use the sine rule to find the missing lengths.

$$\begin{aligned}
\frac{BC}{\sin \hat{A}} &= \frac{AB}{\sin \hat{C}} \\
BC &= \frac{AB\ .\ \sin \hat{A}}{\sin \hat{C}} \\
&= \frac{(0{,}67 \text{ km}) \sin 37°}{\sin 128°} \\
&= 0{,}51 \text{ km}
\end{aligned}$$

$$\begin{aligned}
\frac{AC}{\sin \hat{B}} &= \frac{AB}{\sin \hat{C}} \\
AC &= \frac{AB\ .\ \sin \hat{B}}{\sin \hat{C}} \\
&= \frac{(0{,}67 \text{ km}) \sin 15°}{\sin 128°} \\
&= 0{,}22 \text{ km}
\end{aligned}$$

Exercise 6 – 11: Sine rule

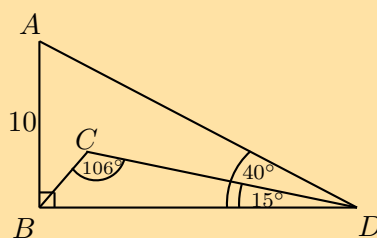
1. Find all the unknown sides and angles of the following triangles:

   a) $\triangle PQR$ in which $\hat{Q} = 64°$; $\hat{R} = 24°$ and $r = 3$
   b) $\triangle KLM$ in which $\hat{K} = 43°$; $\hat{M} = 50°$ and $m = 1$
   c) $\triangle ABC$ in which $\hat{A} = 32{,}7°$; $\hat{C} = 70{,}5°$ and $a = 52{,}3$
   d) $\triangle XYZ$ in which $\hat{X} = 56°$; $\hat{Z} = 40°$ and $x = 50$

2. In $\triangle ABC$, $\hat{A} = 116°$; $\hat{C} = 32°$ and $AC = 23$ m. Find the lengths of the sides AB and BC.

3. In $\triangle RST$, $\hat{R} = 19°$; $\hat{S} = 30°$ and $RT = 120$ km. Find the length of the side ST.

4. In $\triangle KMS$, $\hat{K} = 20°$; $\hat{M} = 100°$ and $s = 23$ cm. Find the length of the side m.

5. In $\triangle ABD$, $\hat{B} = 90°$, $AB = 10$ cm and $A\hat{D}B = 40°$. In $\triangle BCD$, $\hat{C} = 106°$ and $C\hat{D}B = 15°$. Determine BC.

6. In $\triangle ABC$, $\hat{A} = 33°$, $AC = 21$ mm and $AB = 17$ mm. Can you determine BC?

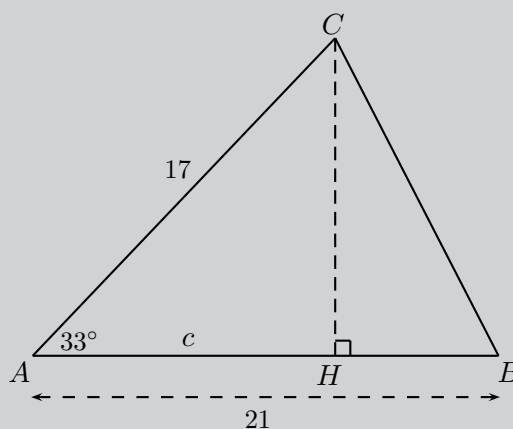
Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 233H 1b. 233J 1c. 233K 1d. 233M 2. 233N 3. 233P
4. 233Q 5. 233R 6. 233S

The cosine rule EMBHS

Investigation: The cosine rule

If a triangle is given with two sides and the included angle known, then we can not solve for the remaining unknown sides and angles using the sine rule. We therefore investigate the cosine rule:

In $\triangle ABC, AB = 21, AC = 17$ and $\hat{A} = 33°$. Find $\hat{B}$.

1. Determine CB:

   a) Construct $CH \perp AB$.

   b) Let $AH = c$ and therefore $HB = \ldots$

c) Applying the theorem of Pythagoras in the right-angled triangles:
In$\triangle CHB$:

$$\begin{aligned} CB^2 &= BH^2 + CH^2 \\ &= (\ldots)^2 + CH^2 \\ &= 21^2 - (2)(21)c + c^2 + CH^2 \ldots\ldots (1) \end{aligned}$$

In $\triangle CHA$:

$$\begin{aligned} CA^2 &= c^2 + CH^2 \\ 17^2 &= c^2 + CH^2 \ldots\ldots (2) \end{aligned}$$

Substitute equation (2) into equation (1):

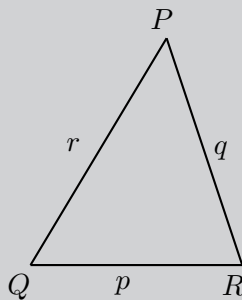
$$CB^2 = 21^2 - (2)(21)c + 17^2$$

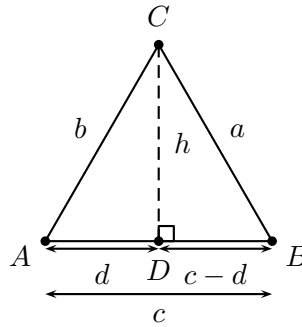
Now c is the only remaining unknown. In $\triangle CHA$:

$$\begin{aligned} \frac{c}{17} &= \cos 33° \\ \therefore c &= 17 \cos 33° \end{aligned}$$

Therefore we have that

$$\begin{aligned} CB^2 &= 21^2 - (2)(21)c + 17^2 \\ &= 21^2 - (2)(21)(17\cos 33°) + 17^2 \\ &= 21^2 + 17^2 - (2)(21)(17)\cos 33° \\ &= 131{,}189\ldots \\ \therefore CB &= 11{,}5 \end{aligned}$$

2. Use your results to write a general formula for the cosine rule given $\triangle PQR$:

The cosine rule relates the length of a side of a triangle to the angle opposite it and the lengths of the other two sides.

Consider $\triangle ABC$ with $CD \perp AB$:

In $\triangle DCB$: $a^2 = (c-d)^2 + h^2$ from the theorem of Pythagoras.

In $\triangle ACD$: $b^2 = d^2 + h^2$ from the theorem of Pythagoras.

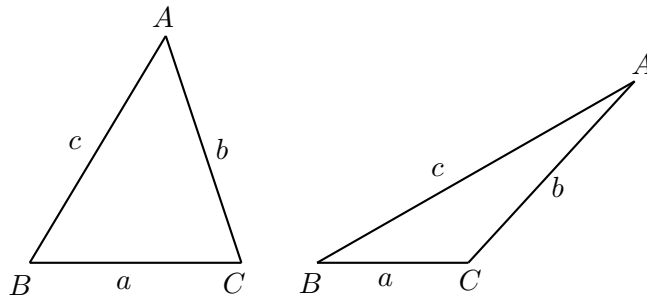
Since h^2 is common to both equations we can write:

$$\begin{aligned} a^2 &= (c-d)^2 + h^2 \\ \therefore h^2 &= a^2 - (c-d)^2 \\ \text{And } b^2 &= d^2 + h^2 \\ \therefore h^2 &= b^2 - d^2 \\ \therefore b^2 - d^2 &= a^2 - (c-d)^2 \\ a^2 &= b^2 + (c^2 - 2cd + d^2) - d^2 \\ &= b^2 + c^2 - 2cd \end{aligned}$$

In order to eliminate d we look at $\triangle ACD$, where we have: $\cos \hat{A} = \frac{d}{b}$. So, $d = b \cos \hat{A}$.

Substituting back we get: $a^2 = b^2 + c^2 - 2bc \cos \hat{A}$.

The cosine rule

In any $\triangle ABC$:

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A}$$

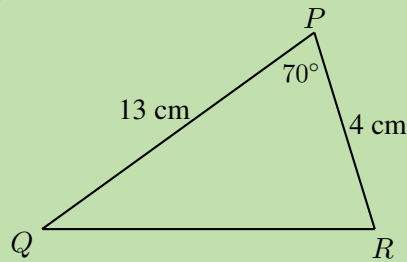
$$b^2 = a^2 + c^2 - 2ac \cos \hat{B}$$

$$c^2 = a^2 + b^2 - 2ab \cos \hat{C}$$

See video: 233T at www.everythingmaths.co.za

Worked example 28: The cosine rule

QUESTION

Determine the length of QR.

SOLUTION

Step 1: Use the cosine rule to solve for the unknown side

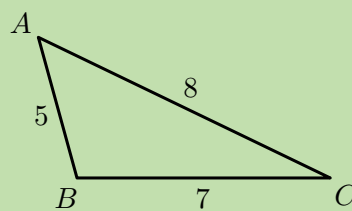
$$\begin{aligned} QR^2 &= PR^2 + QP^2 - 2(PR)(QP)\cos\hat{P} \\ &= 4^2 + 13^2 - 2(4)(13)\cos 70° \\ &= 149{,}42\ldots \\ \therefore QR &= 12{,}2 \end{aligned}$$

Step 2: Write the final answer

$QR = 12{,}2$ cm

Worked example 29: The cosine rule

QUESTION

Determine $\hat{A}$.

SOLUTION

Applying the cosine rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\hat{A} \\ \therefore \cos\hat{A} &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{8^2 + 5^2 - 7^2}{2\,.\,8\,.\,5} \\ &= 0{,}5 \\ \therefore \hat{A} &= 60° \end{aligned}$$

It is very important:

- not to round off before the final answer as this will affect accuracy;
- to take the square root;
- to remember to give units where applicable.

How to determine which rule to use:

1. Area rule:
   - if no perpendicular height is given
2. Sine rule:
   - if no right angle is given
   - if two sides and an angle are given (not the included angle)
   - if two angles and a side are given
3. Cosine rule:
   - if no right angle is given
   - if two sides and the included angle are given
   - if three sides are given

Exercise 6 – 12: The cosine rule

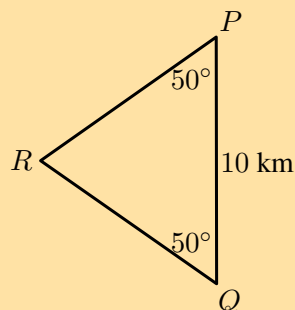
1. Solve the following triangles (that is, find all unknown sides and angles):
   a) $\triangle ABC$ in which $\hat{A} = 70°$; $b = 4$ and $c = 9$
   b) $\triangle RST$ in which $RS = 14$; $ST = 26$ and $RT = 16$
   c) $\triangle KLM$ in which $KL = 5$; $LM = 10$ and $KM = 7$
   d) $\triangle JHK$ in which $\hat{H} = 130°$; $JH = 13$ and $HK = 8$
   e) $\triangle DEF$ in which $d = 4$; $e = 5$ and $f = 7$
2. Find the length of the third side of the $\triangle XYZ$ where:
   a) $\hat{X} = 71{,}4°$; $y = 3{,}42$ km and $z = 4{,}03$ km
   b) $x = 103{,}2$ cm; $\hat{Y} = 20{,}8°$ and $z = 44{,}59$ cm
3. Determine the largest angle in:
   a) $\triangle JHK$ in which $JH = 6$; $HK = 4$ and $JK = 3$
   b) $\triangle PQR$ where $p = 50$; $q = 70$ and $r = 60$

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 233V 1b. 233W 1c. 233X 1d. 233Y 1e. 233Z 2a. 2342
2b. 2343 3a. 2344 3b. 2345

See video: 2346 at www.everythingmaths.co.za

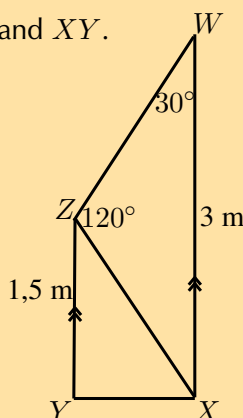
Exercise 6 – 13: Area, sine and cosine rule

1. Q is a ship at a point 10 km due south of another ship P. R is a lighthouse on the coast such that $\hat{P} = \hat{Q} = 50°$.

Determine:

   a) the distance QR

   b) the shortest distance from the lighthouse to the line joining the two ships (PQ).

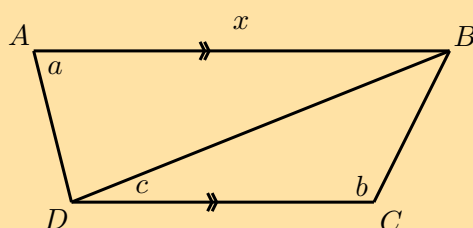
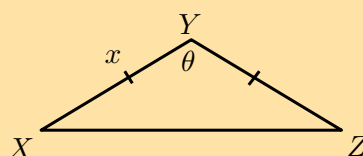
2. $WXYZ$ is a trapezium, $WX \parallel YZ$ with $WX = 3$ m; $YZ = 1{,}5$ m; $\hat{Z} = 120°$ and $\hat{W} = 30°$.

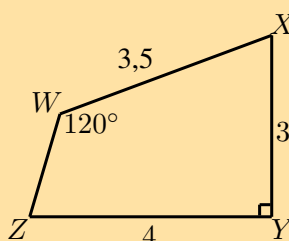
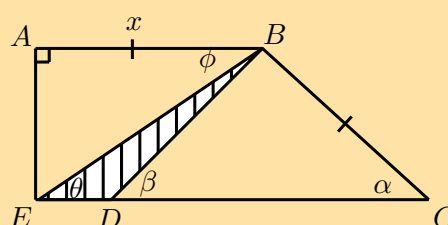
   Determine the distances XZ and XY.

3. On a flight from Johannesburg to Cape Town, the pilot discovers that he has been flying 3° off course. At this point the plane is 500 km from Johannesburg. The direct distance between Cape Town and Johannesburg airports is 1552 km. Determine, to the nearest km:

   a) The distance the plane has to travel to get to Cape Town and hence the extra distance that the plane has had to travel due to the pilot's error.

   b) The correction, to one hundredth of a degree, to the plane's heading (or direction).

4. $ABCD$ is a trapezium (meaning that $AB \parallel CD$). $AB = x$; $B\hat{A}D = a$; $B\hat{C}D = b$ and $B\hat{D}C = c$.

   Find an expression for the length of CD in terms of x, a, b and c.

5. A surveyor is trying to determine the distance between points X and Z. However the distance cannot be determined directly as a ridge lies between the two points. From a point Y which is equidistant from X and Z, he measures the angle $X\hat{Y}Z$.

   a) If $XY = x$ and $X\hat{Y}Z = \theta$, show that $XZ = x\sqrt{2(1 - \cos\theta)}$.
   b) Calculate XZ (to the nearest kilometre) if $x = 240$ km and $\theta = 132°$.

6. Find the area of $WXYZ$ (to two decimal places):

7. Find the area of the shaded triangle in terms of x, α, β, θ and ϕ:

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1. 2347 2. 2348 3. 2349 4. 234B 5. 234C 6. 234D
7. 234F

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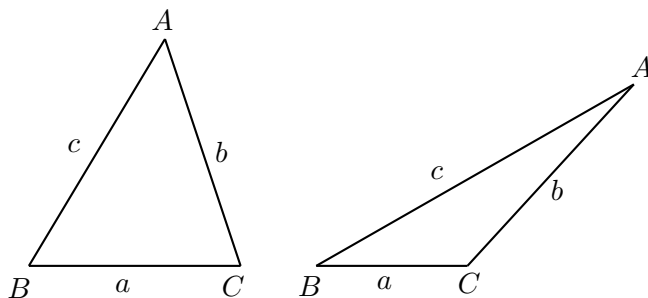
6.6 Summary

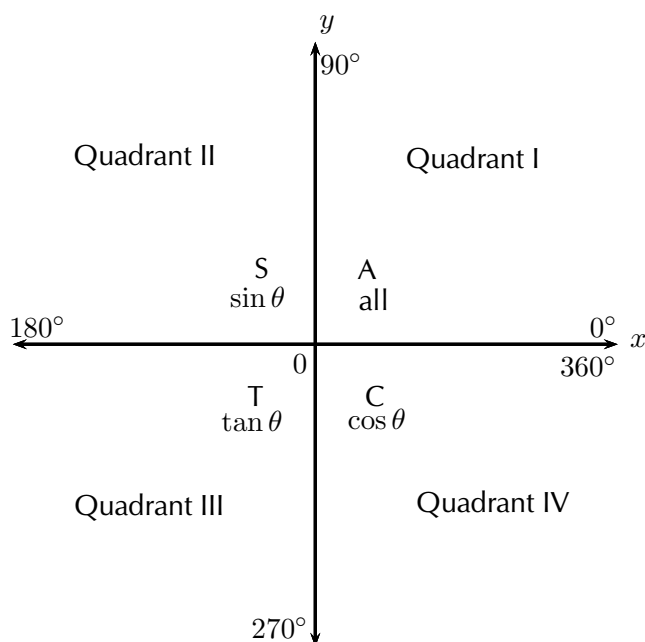
EMBHT

See presentation: 234G at www.everythingmaths.co.za

square identity	quotient identity
$\cos^2\theta + \sin^2\theta = 1$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$

negative angles	periodicity identities	co-function identities
$\sin(-\theta) = -\sin\theta$	$\sin(\theta \pm 360°) = \sin\theta$	$\sin(90° - \theta) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta \pm 360°) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$

sine rule	area rule	cosine rule
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	area $\triangle ABC = \frac{1}{2}bc\sin A$	$a^2 = b^2 + c^2 - 2bc\cos A$
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	area $\triangle ABC = \frac{1}{2}ac\sin B$	$b^2 = a^2 + c^2 - 2ac\cos B$
	area $\triangle ABC = \frac{1}{2}ab\sin C$	$c^2 = a^2 + b^2 - 2ab\cos C$

General solution:

1.

$$\begin{aligned} \text{If } \sin\theta &= x \\ \theta &= \sin^{-1} x + k \;.\; 360° \\ \text{or } \theta &= \left(180° - \sin^{-1} x\right) + k \;.\; 360° \end{aligned}$$

2.

$$\begin{aligned} \text{If } \cos\theta &= x \\ \theta &= \cos^{-1} x + k \;.\; 360° \\ \text{or } \theta &= \left(360° - \cos^{-1} x\right) + k \;.\; 360° \end{aligned}$$

3.

$$\begin{aligned} \text{If } \tan\theta &= x \\ \theta &= \tan^{-1} x + k \;.\; 180° \end{aligned}$$

for $k \in \mathbb{Z}$.

How to determine which rule to use:

1. Area rule:
   - no perpendicular height is given
2. Sine rule:
   - no right angle is given
   - two sides and an angle are given (not the included angle)
   - two angles and a side are given
3. Cosine rule:
   - no right angle is given
   - two sides and the included angle angle are given
   - three sides are given

Exercise 6 – 14: End of chapter exercises

1. Write the following as a single trigonometric ratio:

$$\frac{\cos(90° - A)\sin 20°}{\sin(180° - A)\cos 70°} + \cos(180° + A)\sin(90° + A)$$

2. Determine the value of the following expression without using a calculator:

$$\sin 240° \cos 210° - \tan^2 225° \cos 300° \cos 180°$$

3. Simplify:

$$\frac{\sin(180° + \theta)\sin(\theta + 360°)}{\sin(-\theta)\tan(\theta - 360°)}$$

4. Without the use of a calculator, evaluate:

$$\frac{3\sin 55° \sin^2 325°}{\cos(-145°)} - 3\cos 395° \sin 125°$$

5. Prove the following identities:

    a) $\frac{1}{(\cos x - 1)(\cos x + 1)} = \frac{-1}{\tan^2 x \cos^2 x}$

    b) $(1 - \tan\alpha)\cos\alpha = \sin(90 + \alpha) + \cos(90 + \alpha)$

6. a) Prove: $\tan y + \frac{1}{\tan y} = \frac{1}{\cos^2 y \tan y}$

    b) For which values of $y \in [0°; 360°]$ is the identity above undefined?

7. a) Simplify: $\frac{\sin(180° + \theta)\tan(360° - \theta)}{\sin(-\theta)\tan(180° + \theta)}$

    b) Hence, solve the equation

$$\frac{\sin(180° + \theta)\tan(360° - \theta)}{\sin(-\theta)\tan(180° + \theta)} = \tan\theta$$

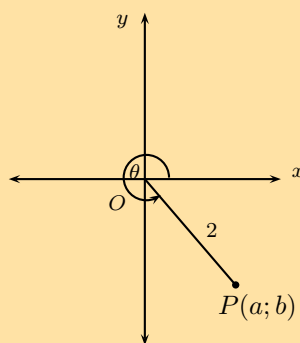
    for $\theta \in [0°; 360°]$.

8. Given $12\tan\theta = 5$ and $\theta > 90°$.

    a) Draw a sketch.

    b) Determine without using a calculator $\sin\theta$ and $\cos(180° + \theta)$.

    c) Use a calculator to find θ (correct to two decimal places).

9.

In the figure, P is a point on the Cartesian plane such that $OP = 2$ units and $\theta = 300°$. Without the use of a calculator, determine:

    a) the values of a and b

    b) the value of $\sin(180° - \theta)$

10. Solve for x with $x \in [-180°; 180°]$ (correct to one decimal place):

    a) $2\sin\frac{x}{2} = 0{,}86$

b) $\tan(x + 10°) = \cos 202{,}6°$

c) $\cos^2 x - 4\sin^2 x = 0$

11. Find the general solution for the following equations:

a) $\frac{1}{2}\sin(x - 25°) = 0{,}25$

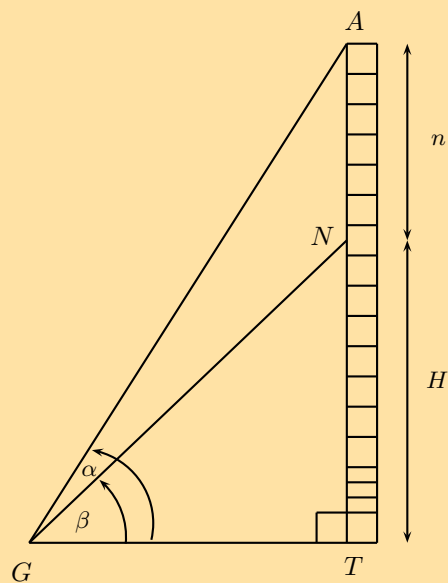
b) $\sin^2 x + 2\cos x = -2$

12. Given the equation: $\sin 2\alpha = 0{,}84$

a) Find the general solution of the equation.

b) Illustrate how this equation could be solved graphically for $\alpha \in [0°; 360°]$.

c) Write down the solutions for $\sin 2\alpha = 0{,}84$ for $\alpha \in [0°; 360°]$.

13.

A is the highest point of a vertical tower AT. At point N on the tower, n metres from the top of the tower, a bird has made its nest. The angle of inclination from G to point A is α and the angle of inclination from G to point N is β.

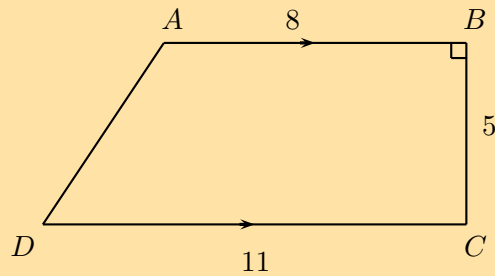
a) Express $A\hat{G}N$ in terms of α and β.

b) Express $\hat{A}$ in terms of α and/or β.

c) Show that the height of the nest from the ground (H) can be determined by the formula

$$H = \frac{n\cos\alpha\sin\beta}{\sin(\alpha - \beta)}$$

d) Calculate the height of the nest H if $n = 10$ m, $\alpha = 68°$ and $\beta = 40°$ (give your answer correct to the nearest metre).

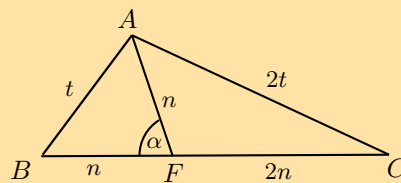
14.

Mr. Collins wants to pave his trapezium-shaped backyard, $ABCD$. $AB \parallel DC$ and $\hat{B} = 90°$. $DC = 11$ m, $AB = 8$ m and $BC = 5$ m.

a) Calculate the length of the diagonal AC.

b) Calculate the length of the side AD.

c) Calculate the area of the patio using geometry.

d) Calculate the area of the patio using trigonometry.

15.

In $\triangle ABC$, $AC = 2A$, $AF = BF$, $A\hat{F}B = \alpha$ and $FC = 2AF$. Prove that $\cos \alpha = \frac{1}{4}$.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 234H 2. 234J 3. 234K 4. 234M 5a. 234N 5b. 234P
6. 234Q 7. 234R 8. 234S 9. 234T 10a. 234V 10b. 234W
10c. 234X 11a. 234Y 11b. 234Z 12. 2352 13. 2353 14. 2354
15. 2355

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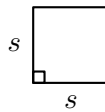
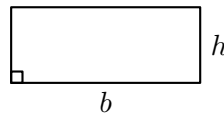
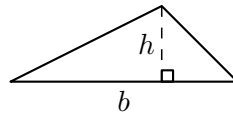
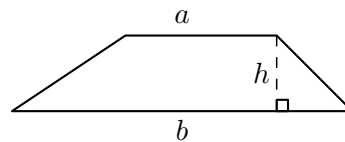
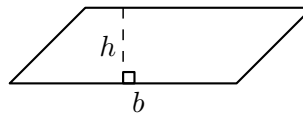
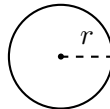
CHAPTER 7

Measurement

7 Measurement

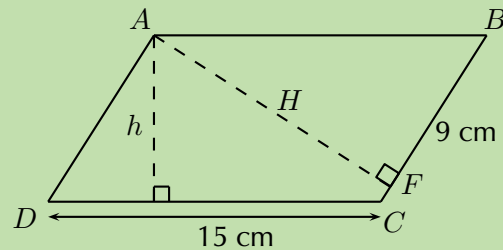
This chapter is a revision of perimeters and areas of two dimensional objects and volumes of three dimensional objects. We also examine different combinations of geometric objects and calculate areas and volumes in a variety of real-life contexts.

See video: 2356 at www.everythingmaths.co.za

7.1 Area of a polygon EMBHV

See video: 2357 at www.everythingmaths.co.za

See video: 2358 at www.everythingmaths.co.za

Worked example 1: Finding the area of a polygon

QUESTION

$ABCD$ is a parallelogram with $DC = 15$ cm, $h = 8$ cm and $BF = 9$ cm.

Calculate:

1. the area of $ABCD$
2. the perimeter of $ABCD$

SOLUTION

Step 1: Determine the area

The area of a parallelogram $ABCD =$ base $\times$ height:

$$\begin{aligned} \text{Area} &= 15 \times 8 \\ &= 120 \text{ cm}^2 \end{aligned}$$

Step 2: Determine the perimeter

The perimeter of a parallelogram $ABCD = 2DC + 2BC$.

To find the length of BC, we use $AF \perp BC$ and the theorem of Pythagoras.

$$\begin{aligned} \text{In } \triangle ABF\text{:} \quad AF^2 &= AB^2 - BF^2 \\ &= 15^2 - 9^2 \\ &= 144 \\ \therefore AF &= 12 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Area} ABCD &= BC \times AF \\ 120 &= BC \times 12 \\ \therefore BC &= 10 \text{ cm} \end{aligned}$$

$$\begin{aligned} \therefore \text{Perimeter} ABCD &= 2(15) + 2(10) \\ &= 50 \text{ cm} \end{aligned}$$

Exercise 7 – 1: Area of a polygon

1. Vuyo and Banele are having a competition to see who can build the best kite using balsa wood (a lightweight wood) and paper. Vuyo decides to make his kite with one diagonal 1 m long and the other diagonal 60 cm long. The intersection of the two diagonals cuts the longer diagonal in the ratio $1 : 3$.

   Banele also uses diagonals of length 60 cm and 1 m, but he designs his kite to be rhombus-shaped.

   a) Draw a sketch of Vuyo's kite and write down all the known measurements.

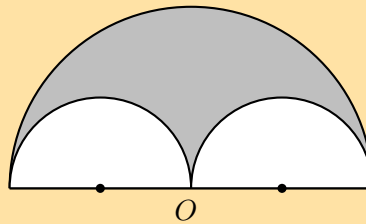
   b) Determine how much balsa wood Vuyo will need to build the outside frame of the kite (give answer correct to the nearest cm).

   c) Calculate how much paper he will need to cover the frame of the kite.

   d) Draw a sketch of Banele's kite and write down all the known measurements.

   e) Determine how much wood and paper Banele will need for his kite.

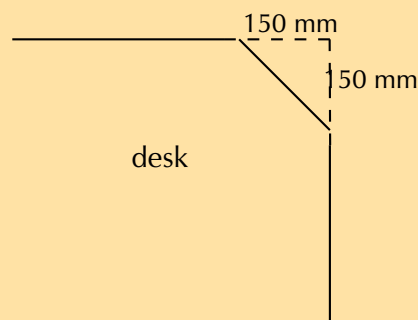
   f) Compare the two designs and comment on the similarities and differences. Which do you think is the better design? Motivate your answer.

2. O is the centre of the bigger semi-circle with a radius of 10 units. Two smaller semi-circles are inscribed into the bigger one, as shown on the diagram. Calculate the following (in terms of π):

   a) The area of the shaded figure.

   b) The perimeter enclosing the shaded area.

3. Karen's engineering textbook is 30 cm long and 20 cm wide. She notices that the dimensions of her desk are in the same proportion as the dimensions of her textbook.

   a) If the desk is 90 cm wide, calculate the area of the top of the desk.

   b) Karen uses some cardboard to cover each corner of her desk with an isosceles triangle, as shown in the diagram:

   Calculate the new perimeter and area of the visible part of the top of her desk.

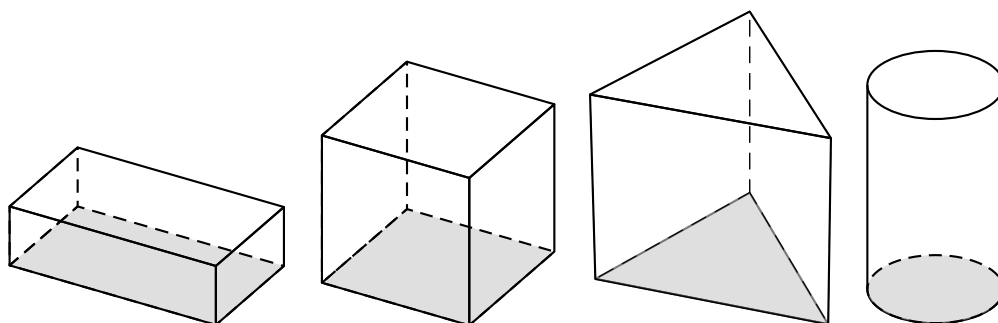
c) Use this new area to calculate the dimensions of a square desk with the same desk top area.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 2359 2. 235B 3. 235C

7.2 Right prisms and cylinders EMBHW

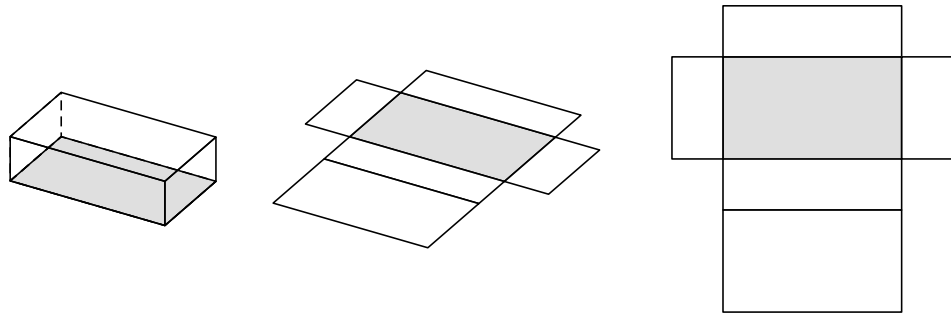
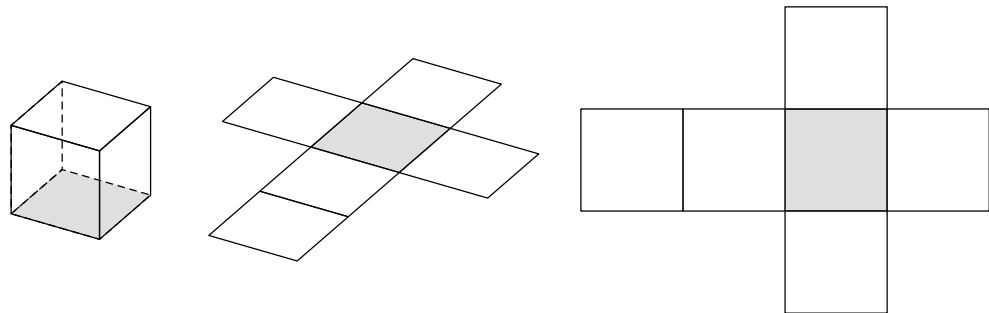
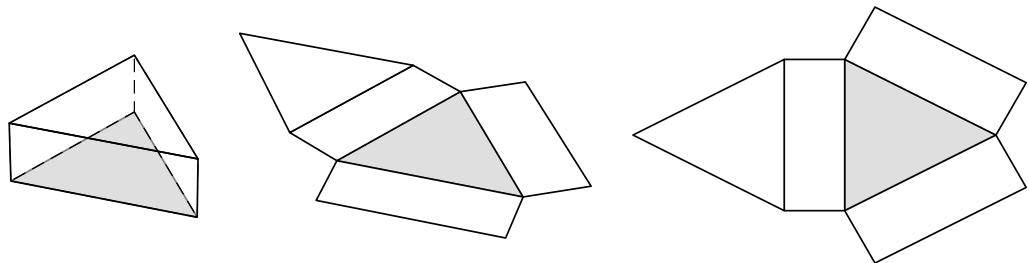
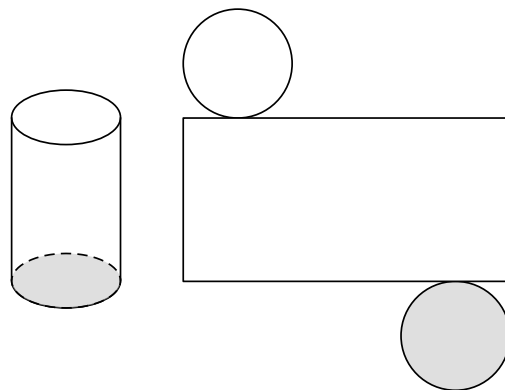
A right prism is a geometric solid that has a polygon as its base and vertical sides perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and sides are right angles.

A triangular prism has a triangle as its base, a rectangular prism has a rectangle as its base, and a cube is a rectangular prism with all its sides of equal length. A cylinder is another type of right prism which has a circle as its base. Examples of right prisms are given below: a rectangular prism, a cube, a triangular prism and a cylinder.

Surface area of prisms and cylinders EMBHX

Surface area is the total area of the exposed or outer surfaces of a prism. This is easier to understand if we imagine the prism to be a cardboard box that we can unfold. A solid that is unfolded like this is called a net. When a prism is unfolded into a net, we can clearly see each of its faces. In order to calculate the surface area of the prism, we can then simply calculate the area of each face, and add them all together.

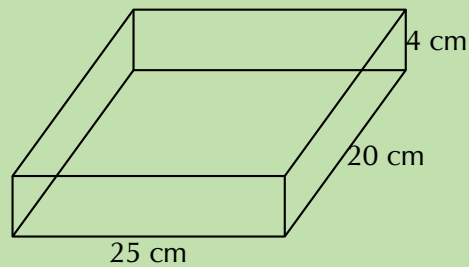
For example, when a triangular prism is unfolded into a net, we can see that it has two faces that are triangles and three faces that are rectangles. To calculate the surface area of the prism, we find the area of each triangle and each rectangle, and add them together.

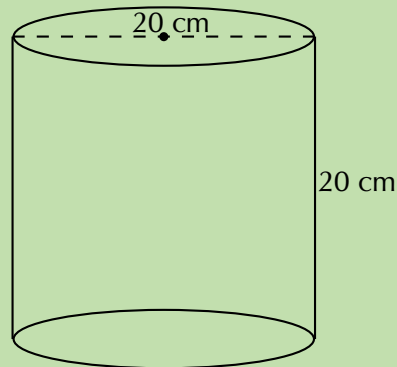
In the case of a cylinder the top and bottom faces are circles and the curved surface flattens into a rectangle with a length that is equal to the circumference of the circular base. To calculate the surface area we therefore find the area of the two circles and the rectangle and add them together.

Below are examples of right prisms that have been unfolded into nets. A rectangular prism unfolded into a net is made up of six rectangles.

A cube unfolded into a net is made up of six identical squares.

A triangular prism unfolded into a net is made up of two triangles and three rectangles. The sum of the lengths of the rectangles is equal to the perimeter of the triangles.

A cylinder unfolded into a net is made up of two identical circles and a rectangle with length equal to the circumference of the circles.

Worked example 2: Calculating surface area

QUESTION

A box of chocolates has the following dimensions:

$$\text{length} = 25 \text{ cm}$$
$$\text{width} = 20 \text{ cm}$$
$$\text{height} = 4 \text{ cm}$$

And a cylindrical tin of biscuits has the following dimensions:

$$\text{diameter} = 20 \text{ cm}$$
$$\text{height} = 20 \text{ cm}$$

1. Calculate the area of the wrapping paper needed to cover the entire box (assume no overlapping at the corners).
2. Determine if this same sheet of wrapping paper would be enough to cover the tin of biscuits.

SOLUTION

Step 1: Determine the area of the rectangular box

$$\begin{aligned}\text{Surface area} &= 2 \times (25 \times 20) + 2 \times (20 \times 4) + 2 \times (25 \times 4) \\ &= 1360 \text{ cm}^2\end{aligned}$$

Step 2: Determine the area of the cylindrical tin

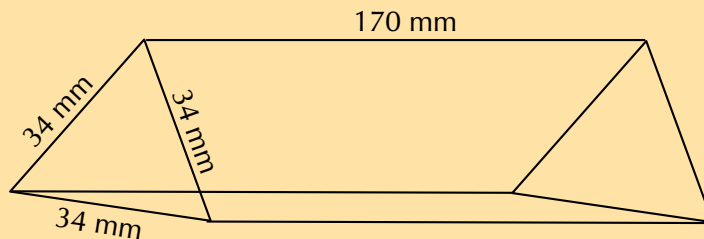
The radius of the cylinder $= \frac{20}{2} = 10$ cm.

$$\begin{aligned}\text{Surface area} &= 2 \times \pi(10)^2 + 2\pi(10)(20) \\ &= 1885 \text{ cm}^2\end{aligned}$$

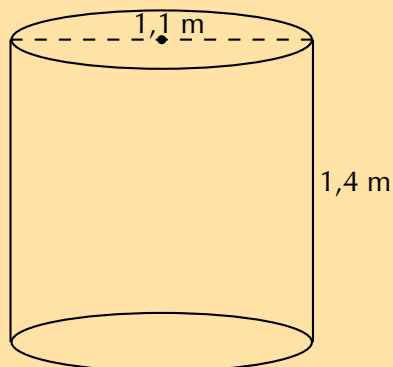
Step 3: Write the final answer

No, the area of the sheet of wrapping paper used to cover the box is not big enough to cover the tin.

Exercise 7 – 2: Calculating surface area

1. A popular chocolate container is an equilateral right triangular prism with sides of 34 mm. The box is 170 mm long. Calculate the surface area of the box (to the nearest square centimetre).

2. Gordon buys a cylindrical water tank to catch rain water off his roof. He discovers a full $2\ \ell$ tin of green paint in his garage and decides to paint the tank (not the base). If he uses 250 ml to cover 1 m^2, will he have enough green paint to cover the tank with one layer of paint?

   Dimensions of the tank:

$$\begin{aligned}\text{diameter} &= 1{,}1 \text{ m} \\ \text{height} &= 1{,}4 \text{ m}\end{aligned}$$

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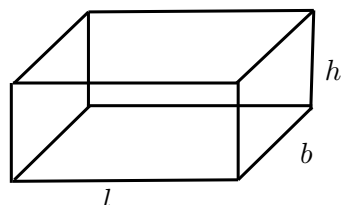
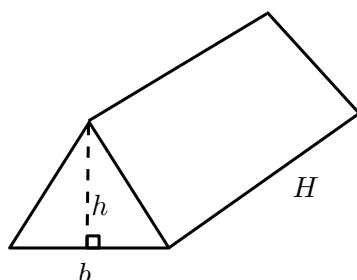
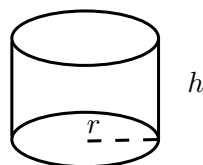
1. 235D 2. 235F

Volume of prisms and cylinders EMBHY

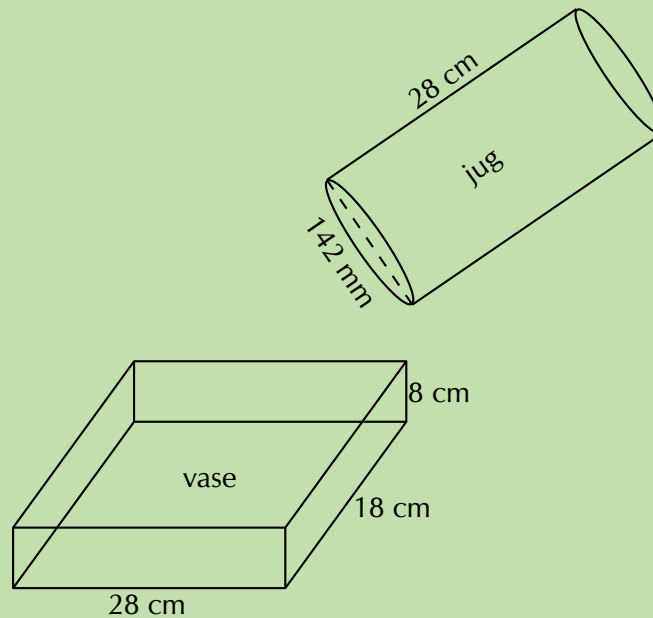
Volume, sometimes also called capacity, is the three dimensional space occupied by an object, or the contents of an object. It is measured in cubic units.

The volume of a right prism is simply calculated by multiplying the area of the base of a solid by the height of the solid.

$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= \text{area of rectangle} \times \text{height} \\ &= l \times b \times h\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= \text{area of triangle} \times \text{height} \\ &= \left(\frac{1}{2}b \times h\right) \times H\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= \text{area of circle} \times \text{height} \\ &= \pi r^2 \times h\end{aligned}$$

See video: 235G at www.everythingmaths.co.za

Worked example 3: Calculating volume

QUESTION

A rectangular glass vase with dimensions 28 cm × 18 cm × 8 cm is used for flower arrangements. A florist uses a platic cylindrical jug to pour water into the glass vase. The jug has a diameter of 142 mm and a height of 28 cm.

1. Will the plastic jug hold 5 ℓ of water?
2. Will a full jug of water be enough to fill the glass vase?

SOLUTION

Step 1: Determine the volume of the plastic jug

The diameter of the jug is 142 mm, therefore the radius $= \frac{142}{2\times 10} = 7{,}1$ cm.

Volume of a cylinder = area of the base × height

$$\begin{aligned} \text{Volume of the jug} &= \pi r^2 \times h \\ &= \pi \times (7{,}1)^2 \times 28 \\ &= 4434 \text{ cm}^3 \\ \text{And } 1000 \text{ cm}^3 &= 1\ \ell \\ \therefore \text{Volume of the jug} &= \frac{4434}{1000} \\ &= 4{,}434\ \ell \end{aligned}$$

No, the capacity of the jug is not enough to hold 5 ℓ of water.

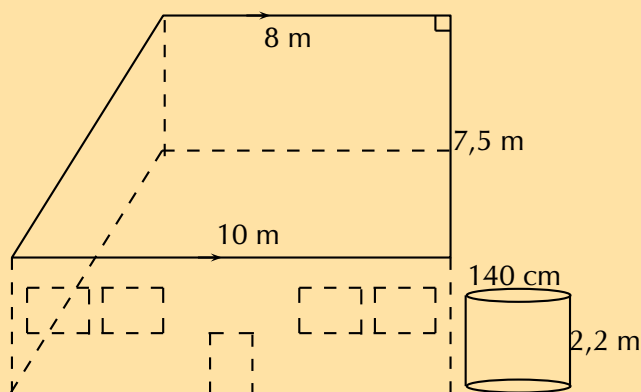
Step 2: Determine the volume of the glass vase

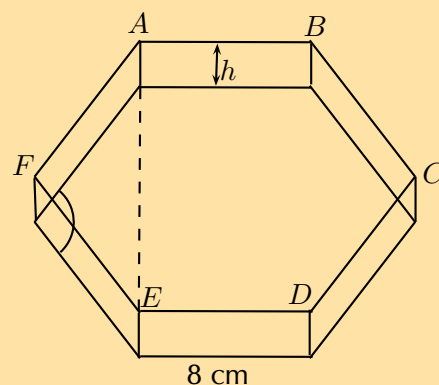
Volume of a rectangular prism = area of the base × height

$$\begin{aligned}\text{Volume of the vase} &= l \times b \times h \\ &= 28 \times 18 \times 8 \\ &= 4032 \text{ cm}^3 \\ \therefore \text{Volume of the vase} &= \frac{4032}{1000} \\ &= 4{,}032\ \ell\end{aligned}$$

Yes, the volume of the jug is greater than the volume of the vase.

Exercise 7 – 3: Calculating volume

1. The roof of Phumza's house is the shape of a right-angled trapezium. A cylindrical water tank is positioned next to the house so that the rain on the roof runs into the tank. The diameter of the tank is 140 cm and the height is 2,2 m.

   a) Determine the area of the roof.
   b) Determine how many litres of water the tank can hold.

2. The length of a side of a hexagonal sweet tin is 8 cm and its height is equal to half of the side length.

   a) Show that the interior angles are equal to $120°$.

b) Determine the length of the line AE.

c) Calculate the volume of the tin.

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1. 235H 2. 235J

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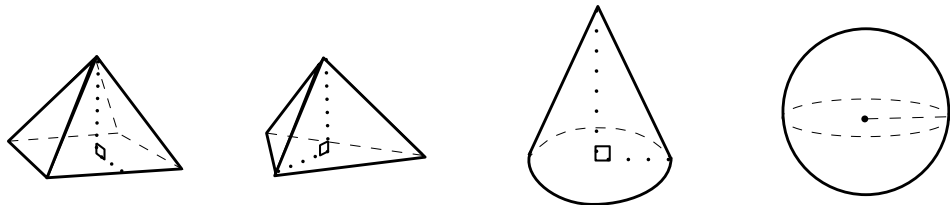
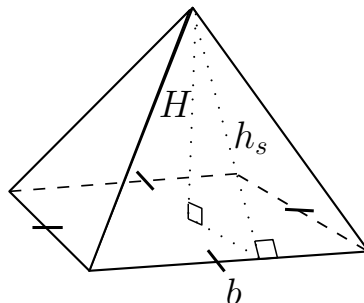
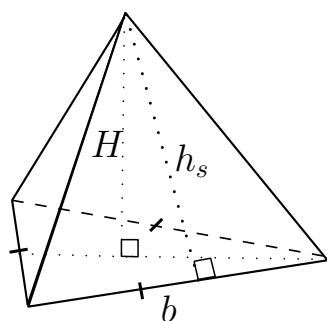
7.3 Right pyramids, right cones and spheres EMBHZ

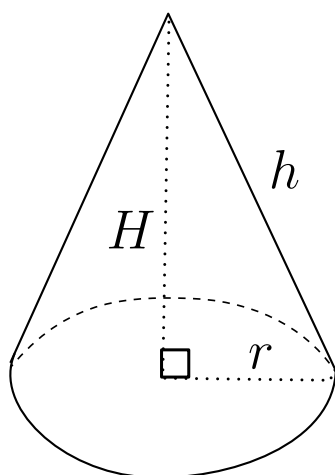
A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. In other words the sides are **not** perpendicular to the base.

The triangular pyramid and square pyramid take their names from the shape of their base. We call a pyramid a "right pyramid" if the line between the apex and the centre of the base is perpendicular to the base. Cones are similar to pyramids except that their bases are circles instead of polygons. Spheres are solids that are perfectly round and look the same from any direction.

Surface area of pyramids, cones and spheres EMBJ2

$$\begin{aligned}\text{Surface area} &= \text{area of base} + \\ &\quad \text{area of triangular sides} \\ &= b^2 + 4\left(\tfrac{1}{2}bh_s\right) \\ &= b\left(b + 2h_s\right)\end{aligned}$$

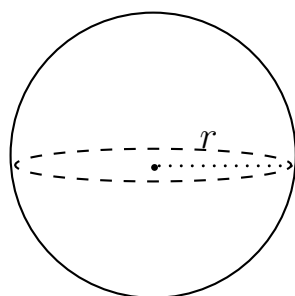
Triangular pyramid

Surface area = area of base + area of triangular sides

$= \left(\frac{1}{2}b \times h_b\right) + 3\left(\frac{1}{2}b \times h_s\right)$

$= \frac{1}{2}b\left(h_b + 3h_s\right)$

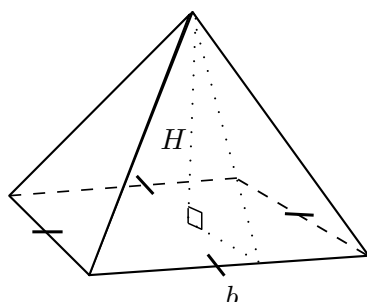
Right cone

Surface area = area of base + area of walls

$= \pi r^2 + \frac{1}{2} \times 2\pi rh$

$= \pi r\left(r + h\right)$

Sphere

Surface area $= 4\pi r^2$

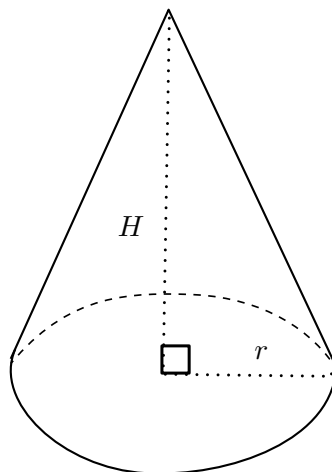
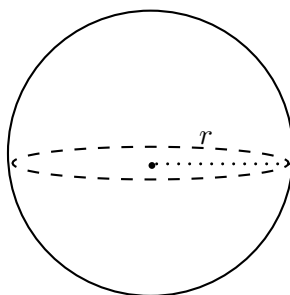
Volume of pyramids, cones and spheres EMBJ3

Square pyramid

Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid

$= \frac{1}{3} \times b^2 \times H$

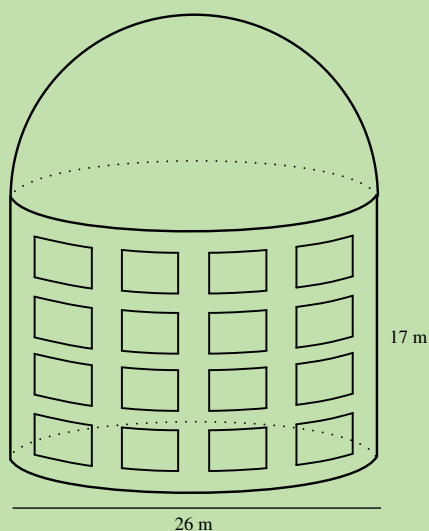
Triangular pyramid

Volume $= \frac{1}{3} \times$ area of base $\times$ height of pyramid

$= \frac{1}{3} \times \frac{1}{2}bh \times H$

See video: 235K at www.everythingmaths.co.za

Worked example 4: Finding surface area and volume

QUESTION

The Southern African Large Telescope (SALT) is housed in a cylindrical building with a domed roof in the shape of a hemisphere. The height of the building wall is 17 m and the diameter is 26 m.

1. Calculate the total surface area of the building.
2. Calculate the total volume of the building.

SOLUTION

Step 1: Calculate the total surface area

$$\text{Total surface area} = \text{area of the dome} + \text{area of the cylinder}$$

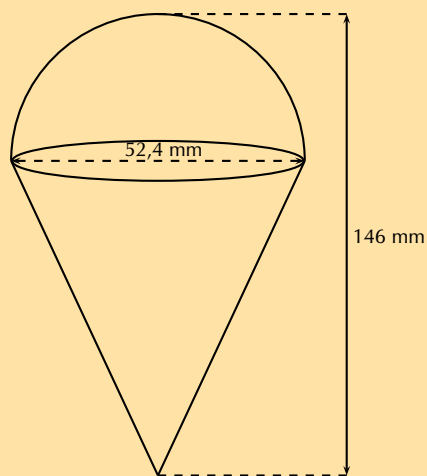
$$\begin{aligned}\text{Surface area} &= \left[\frac{1}{2}(4\pi r^2)\right] + [2\pi r \times h] \\ &= \frac{1}{2}(4\pi)(13)^2 + 2\pi(13)(17) \\ &= 2450 \text{ m}^2\end{aligned}$$

Step 2: Calculate the total volume

$$\text{Total volume} = \text{volume of the dome} + \text{volume of the cylinder}$$

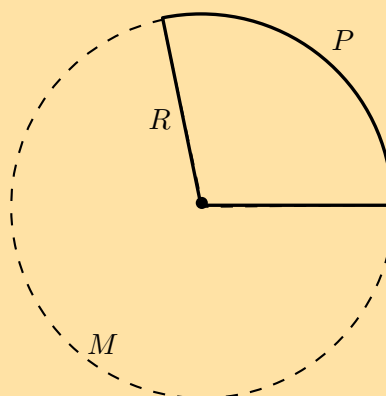
$$\begin{aligned}\text{Volume} &= \left[\frac{1}{2} \times \left(\frac{4}{3}\pi r^3\right)\right] + \left[\pi r^2 h\right] \\ &= \frac{2}{3}\pi(13)^3 + \pi(11)^2(13) \\ &= 9543 \text{ m}^3\end{aligned}$$

Exercise 7 – 4: Finding surface area and volume

1. An ice-cream cone has a diameter of $52{,}4$ mm and a total height of 146 mm.

a) Calculate the surface area of the ice-cream and the cone.

b) Calculate the total volume of the ice-cream and the cone.

c) How many ice-cream cones can be made from a $5\ \ell$ tub of ice-cream (assume the cone is completely filled with ice-cream)?

d) Consider the net of the cone given below. R is the length from the tip of the cone to its perimeter, P.

i. Determine the value of R.
ii. Calculate the length of arc P.
iii. Determine the length of arc M.

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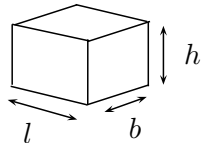
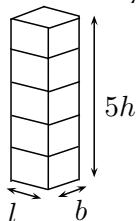
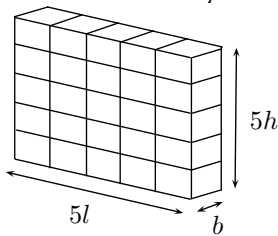
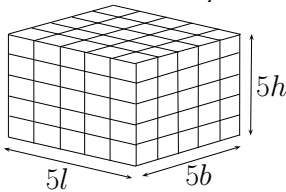
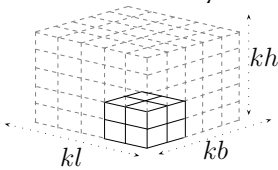
1. 235M

7.4 Multiplying a dimension by a constant factor EMBJ4

When one or more of the dimensions of a prism or cylinder is multiplied by a constant, the surface area and volume will change. The new surface area and volume can be calculated by using the formulae from the preceding section.

It is important to see a relationship between the change in dimensions and the resulting change in surface area and volume. These relationships make it simpler to calculate the new volume or surface area of an object when its dimensions are scaled up or down.

Consider a rectangular prism of dimensions l, b and h. Below we multiply one, two and three of its dimensions by a constant factor of 5 and calculate the new volume and surface area.

Dimensions	Volume	Surface
Original dimensions (figure: l, b, h)	$V = l \times b \times h$ $= lbh$	A $= 2\left[(l \times h) + (l \times b) + (b \times h)\right]$ $= 2\left(lh + lb + bh\right)$
Multiply one dimension by 5 (figure: l, b, $5h$)	$V_1 = l \times b \times 5h$ $= 5\left(lbh\right)$ $= 5V$	A_1 $= 2\left[(l \times 5h) + (l \times b) + (b \times 5h)\right]$ $= 2\left(5lh + lb + 5bh\right)$
Multiply two dimensions by 5 (figure: $5l$, b, $5h$)	$V = 5l \times b \times 5h$ $= 5\ .\ 5(lbh)$ $= 5^2V$	A_2 $= 2\left[(5l \times 5h) + (5l \times b) + (b \times 5h)\right]$ $= 2 \times 5(5lh + lb + bh)$
Multiply all three dimensions by 5 (figure: $5l$, $5b$, $5h$)	$V = 5l \times 5b \times 5h$ $= 5^3(lbh)$ $= 5^3V$	A_3 $= 2\left[(5l \times 5h) + (5l \times 5b) + (5b \times 5h)\right]$ $= 2 \times (5^2lh + 5^2lb + 5^2bh)$ $= 5^2 \times 2(lh + lb + bh)$ $= 5^2A$
Multiply all three dimensions by k (figure: kl, kb, kh)	$V = kl \times kb \times kh$ $= k^3(lbh)$ $= k^3V$	A_k $= 2\left[(kl \times kh) + (kl \times kb) + (kb \times kh)\right]$ $= 2 \times (k^2lh + k^2lb + k^2bh)$ $= k^2 \times 2(lh + lb + bh)$ $= k^2A$

Worked example 5: The effects of k

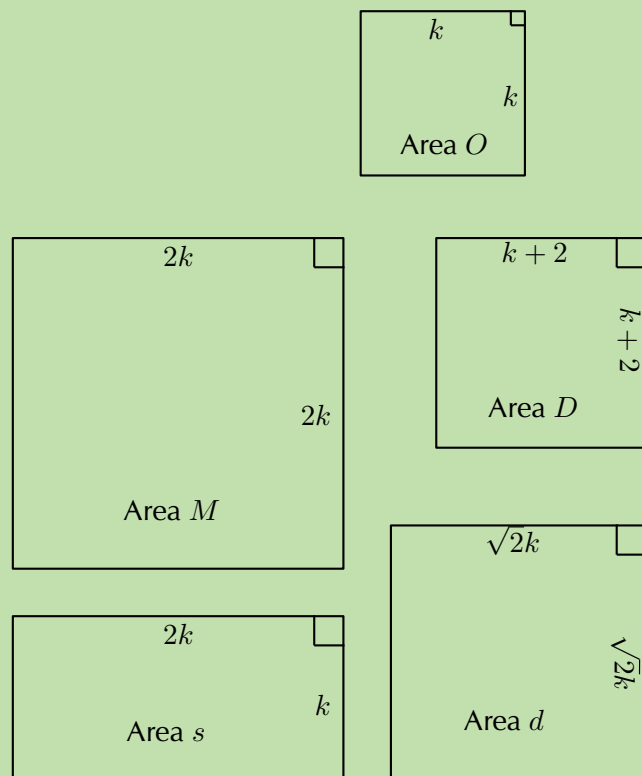
QUESTION

The Nash family wants to build a television room onto their house. The dad draws up the plans for the new square room of length k metres. The mum looks at the plans and decides that the area of the room needs to be doubled. To achieve this:

- the mum suggests doubling the length of the sides of the room
- the dad recommends adding 2 m to the length of the sides
- the daughter suggests multiplying the length of the sides by a factor of $\sqrt{2}$
- the son suggests doubling only the width of the room

Who's suggestion will double the area of the square room? Show all calculations.

SOLUTION

Step 1: Draw a sketch

Step 2: Calculate and compare

First calculate the area of the square room in the original plan:

$$\begin{aligned} \text{Area } O &= \text{length} \times \text{length} \\ &= k^2 \end{aligned}$$

Therefore, double the area of the room would be $2k^2$.

Consider the mum's suggestion of doubling the length of the sides of the room:

$$\begin{aligned}\text{Area } M &= \text{length} \times \text{length}\\ &= 2k \times 2k\\ &= 4k^2\end{aligned}$$

This area would be 4 times the original area.

The dad suggests adding 2 m to the length of the sides of the room:

$$\begin{aligned}\text{Area } D &= \text{length} \times \text{length}\\ &= (k+2) \times (k+2)\\ &= k^2 + 4k + 2\\ &\neq 2k^2\end{aligned}$$

This is not double the original area.

The daughter suggests multiplying the length of the sides by a factor of $\sqrt{2}$:

$$\begin{aligned}\text{Area } d &= \text{length} \times \text{length}\\ &= \sqrt{2}k \times \sqrt{2}k\\ &= 2k^2\end{aligned}$$

The daughter's suggestion would double the area of the room. Practically, the length of the room could be multiplied by $\sqrt{2} \approx 1{,}41$ which would given an area of $1{,}96$ m^2.

The son suggests doubling only the width of the room:

$$\begin{aligned}\text{Area } s &= \text{length} \times \text{length}\\ &= 2k \times k\\ &= 2k^2\end{aligned}$$

The son's suggestion would double the area of the room, however the room would no longer be a square.

Step 3: Write the final answer

The daughter's suggestion of multiplying the length of the sides of the room by a factor of $\sqrt{2}$ would keep the shape of the room a square and would double the area of the room.

Exercise 7 – 5: The effects of k

1. Complete the following sentences:

   a) If one dimension of a cube is multiplied by a factor $\frac{1}{2}$, the volume of the cube ...

   b) If two dimensions of a cube are multiplied by a factor 7, the volume of the cube ...

c) If three dimensions of a cube are multiplied by a factor 3, then:

i. each side of the cube will ...

ii. the outer surface area of the cube will ...

iii. the volume of the cube will ...

d) If each side of a cube is halved, then:

i. the outer surface area of the cube will ...

ii. the volume of the cube will ...

2. The municipality intends building a swimming pool of volume W^3 cubic metres. However, they realise that it will be very expensive to fill the pool with water, so they decide to make the pool smaller.

a) The length and breadth of the pool are reduced by a factor of $\frac{7}{10}$. Express the new volume in terms of W.

b) The dimensions of the pool are reduced so that the volume of the pool decreases by a factor of $0{,}8$. Determine the new dimensions of the pool in terms of W (remember that the pool must be a cube).

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1. 235N 2. 235P

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7.5 Summary EMBJ5

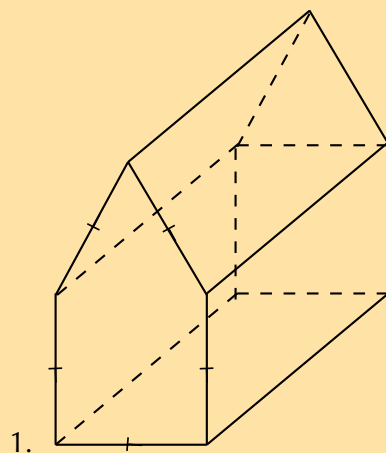
See presentation: 235Q at www.everythingmaths.co.za

1. Area is the two dimensional space inside the boundary of a flat object.
2. Area formulae:
   - square: s^2
   - rectangle: $b \times h$
   - triangle: $\frac{1}{2}b \times h$
   - trapezium: $\frac{1}{2}(a+b) \times h$
   - parallelogram: $b \times h$
   - circle: πr^2
3. Surface area is the total area of the exposed or outer surfaces of a prism.
4. A net is the unfolded "plan" of a solid.
5. Volume is the three dimensional space occupied by an object, or the contents of an object.
   - Volume of a rectangular prism: $l \times b \times h$

- Volume of a triangular prism: $\left(\frac{1}{2}b \times h\right) \times H$
- Volume of a square prism or cube: s^3
- Volume of a cylinder: $\pi r^2 \times h$

6. A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. The sides are not perpendicular to the base.
7. Surface area formulae:
   - square pyramid: $b\,(b + 2h)$
   - triangular pyramid: $\frac{1}{2}b\,(h_b + 3h_s)$
   - right cone: $\pi r\,(r + h_s)$
   - sphere: $4\pi r^2$
8. Volume formulae:
   - square pyramid: $\frac{1}{3} \times b^2 \times H$
   - triangular pyramid: $\frac{1}{3} \times \frac{1}{2}bh \times H$
   - right cone: $\frac{1}{3} \times \pi r^2 \times H$
   - sphere: $\frac{4}{3}\pi r^3$

Exercise 7 – 6: End of chapter exercises

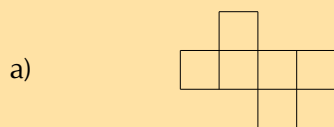
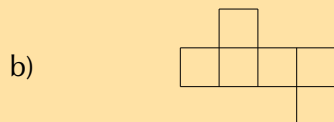
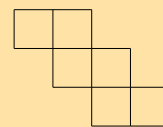
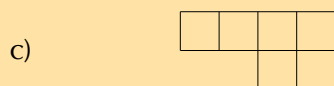
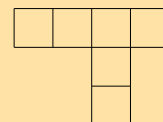
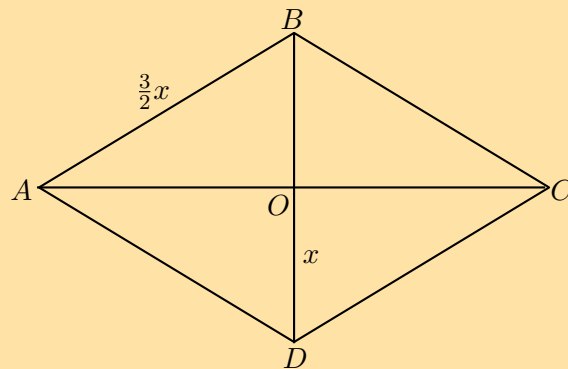
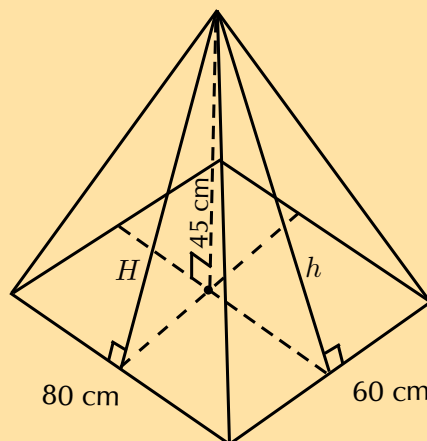
1.

   a) Describe this figure in terms of a prism.

   b) Draw a net of this figure.

2. Which of the following is a net of a cube?

a)

b)

c)

d)

e)

3. Name and draw the following figures:
   a) A prism with the least number of sides.
   b) A pyramid with the least number of vertices.
   c) A right prism with a kite base.
4. a) i. Determine how much paper is needed to make a box of width 16 cm, height 3 cm and length 20 cm (assume no overlapping at corners).
      ii. Give a mathematical name for the shape of the box.
      iii. Calculate the volume of the box.
   b) Determine how much paper is needed to make a cube with a capacity of $1\ \ell$.
   c) Compare the box and the cube. Which has the greater volume and which requires the most paper to make?
5. $ABCD$ is a rhombus with sides of length $\frac{3}{2}x$ millimetres. The diagonals intersect at O and length $DO = x$ millimetres. Express the area of $ABCD$ in terms of x.

6. The diagram shows a rectangular pyramid with a base of length 80 cm and breadth 60 cm. The vertical height of the pyramid is 45 cm.

   a) Calculate the volume of the pyramid.
   b) Calculate H and h.
   c) Calculate the surface area of the pyramid.
7. A group of children are playing soccer in a field. The soccer ball has a capacity of 5000 cc (cubic centimetres). A drain pipe in the corner of the field has a diameter of 20 cm. Is it possible for the children to lose their ball down the pipe? Show your calculations.

8. A litre of washing powder goes into a standard cubic container at the factory.

   a) Determine the length of the sides of the container.

   b) Determine the dimensions of the cubic container required to hold double the volume of washing powder.

9. A cube has sides of length k units.

   a) Describe the effect on the volume of the cube if the height is tripled.

   b) If all three dimensions of the cube are tripled, determine the effect on the outer surface area.

   c) If all three dimensions of the cube are tripled, determine the effect on the volume.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 235R 2. 235S 3a. 235T 3b. 235V 3c. 235W 4. 235X
5. 235Y 6. 235Z 7. 2362 8. 2363 9. 2364

CHAPTER

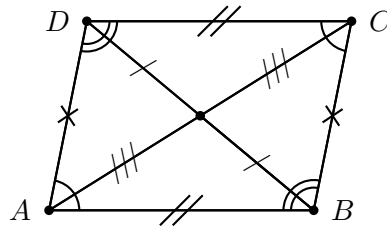
Euclidean geometry

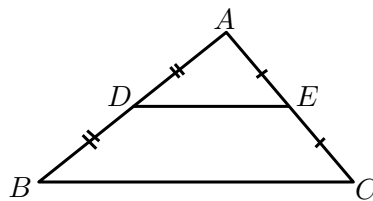
8.1 Revision EMBJ6

Parallelogram EMBJ7

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.

Summary of the properties of a parallelogram:

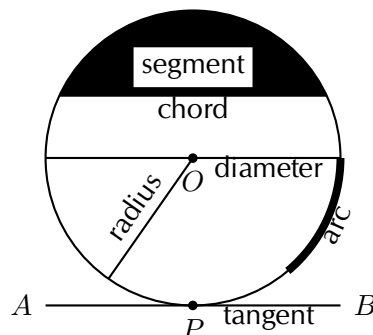
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

The mid-point theorem EMBJ8

The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

Given: $AD = DB$ and $AE = EC$, we can conclude that $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

8.2 Circle geometry

Terminology

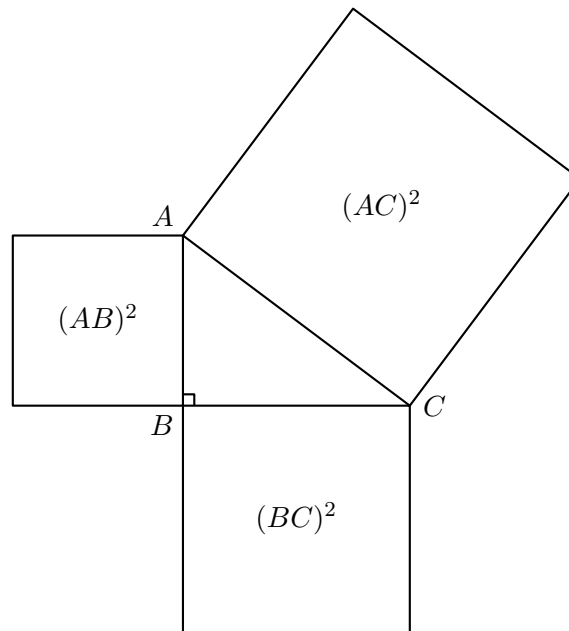
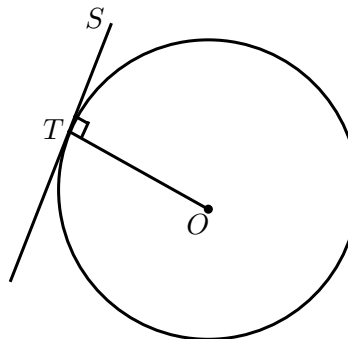
The following terms are regularly used when referring to circles:

- **Arc** — a portion of the circumference of a circle.
- **Chord** — a straight line joining the ends of an arc.
- **Circumference** — the perimeter or boundary line of a circle.
- **Radius** (r) — any straight line from the centre of the circle to a point on the circumference.
- **Diameter** — a special chord that passes through the centre of the circle. A diameter is a straight line segment from one point on the circumference to another point on the circumference that passes through the centre of the circle.
- **Segment** — part of the circle that is cut off by a chord. A chord divides a circle into two segments.
- **Tangent** — a straight line that makes contact with a circle at only one point on the circumference.

See video: 2365 at www.everythingmaths.co.za

Axioms

An axiom is an established or accepted principle. For this section, the following are accepted as axioms.

1. The theorem of Pythagoras states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.

$$(AC)^2 = (AB)^2 + (BC)^2$$

2. A tangent is perpendicular to the radius ($OT \perp ST$), drawn at the point of contact with the circle.

Theorems

EMBJB

A theorem is a hypothesis (proposition) that can be shown to be true by accepted mathematical operations and arguments. A proof is the process of showing a theorem to be correct.

The converse of a theorem is the reverse of the hypothesis and the conclusion. For example, given the theorem "if A, then B", the converse is "if B, then A".

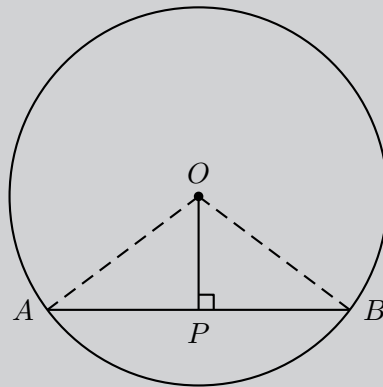
Theorem: Perpendicular line from circle centre bisects chord

STATEMENT

If a line is drawn from the centre of a circle perpendicular to a chord, then it bisects the chord.

(Reason: $\perp$ from centre bisects chord)

Given:

Circle with centre O and line OP perpendicular to chord AB.

Required to prove:

$AP = PB$

PROOF

Draw OA and OB.

In $\triangle OPA$ and in $\triangle OPB$,

$$\begin{aligned} OA^2 &= OP^2 + AP^2 && \text{(Pythagoras)} \\ OB^2 &= OP^2 + BP^2 && \text{(Pythagoras)} \end{aligned}$$

and

$$\begin{aligned} OA &= OB && \text{(equal radii)} \\ \therefore AP^2 &= BP^2 \\ \therefore AP &= BP \end{aligned}$$

Therefore OP bisects AB.

Alternative proof:

In $\triangle OPA$ and in $\triangle OPB$,

$$\begin{aligned} O\hat{P}A &= O\hat{P}B && \text{(given } OP \perp AB\text{)} \\ OA &= OB && \text{(equal radii)} \\ OP &= OP && \text{(common side)} \\ \therefore \triangle OPA &\equiv \triangle OPB && \text{(RHS)} \\ \therefore AP &= PB \end{aligned}$$

Therefore OP bisects AB.

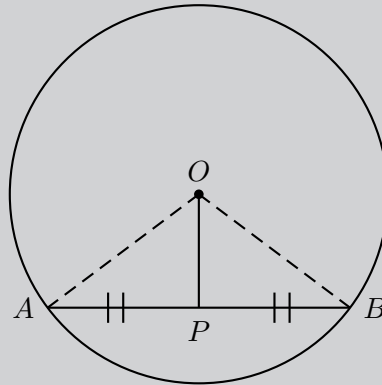
(PROOF NOT FOR EXAMS) Converse: Line from circle centre to mid-point of chord is perpendicular

STATEMENT

If a line is drawn from the centre of a circle to the mid-point of a chord, then the line is perpendicular to the chord.

(Reason: line from centre to mid-point $\perp$)

Given:

Circle with centre O and line OP to mid-point P on chord AB.

Required to prove:

$OP \perp AB$

PROOF

Draw OA and OB.

In $\triangle OPA$ and in $\triangle OPB$,

$$
\begin{aligned}
OA &= OB && \text{(equal radii)} \\
AP &= PB && \text{(given)} \\
OP &= OP && \text{(common side)} \\
\therefore \triangle OPA &\equiv \triangle OPB && \text{(SSS)} \\
\therefore O\hat{P}A &= O\hat{P}B && \\
\text{and } O\hat{P}A + O\hat{P}B &= 180^\circ && (\angle \text{ on str. line}) \\
\therefore O\hat{P}A = O\hat{P}B &= 90^\circ &&
\end{aligned}
$$

Therefore $OP \perp AB$.

See video: 2366 at www.everythingmaths.co.za

Theorem: Perpendicular bisector of chord passes through circle centre

STATEMENT

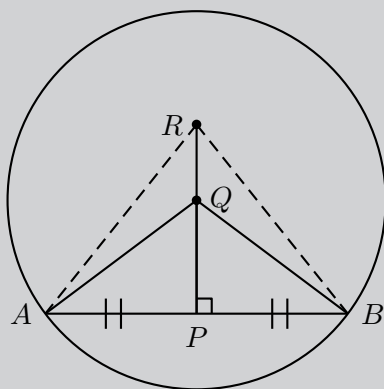
If the perpendicular bisector of a chord is drawn, then the line will pass through the centre of the circle.

(Reason: $\perp$ bisector through centre)

Given:

Circle with mid-point P on chord AB.

Line QP is drawn such that $Q\hat{P}A = Q\hat{P}B = 90°$.

Line RP is drawn such that $R\hat{P}A = R\hat{P}B = 90°$.

Required to prove:

Circle centre O lies on the line PR

PROOF

Draw lines QA and QB.

Draw lines RA and RB.

In $\triangle QPA$ and in $\triangle QPB$,

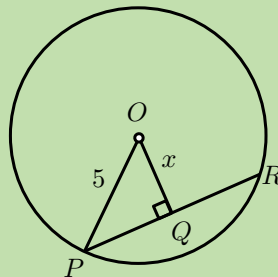
$$\begin{aligned} AP &= PB && \text{(given)} \\ QP &= QP && \text{(common side)} \\ Q\hat{P}A = Q\hat{P}B &= 90° && \text{(given)} \\ \therefore \triangle QPA &\equiv \triangle QPB && \text{(SAS)} \\ \therefore QA &= QB \end{aligned}$$

Similarly it can be shown that in $\triangle RPA$ and in $\triangle RPB$, $RA = RB$.

We conclude that all the points that are equidistant from A and B will lie on the line PR extended. Therefore the centre O, which is equidistant to all points on the circumference, must also lie on the line PR.

Worked example 1: Perpendicular line from circle centre bisects chord

QUESTION

Given $OQ \perp PR$ and $PR = 8$ units, determine the value of x.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles and sides on the diagram

$PQ = QR = 4$ ($\perp$ from centre bisects chord)

Step 2: Solve for x

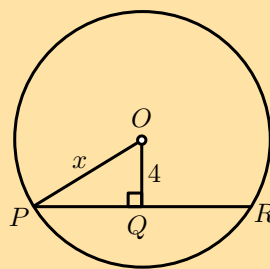
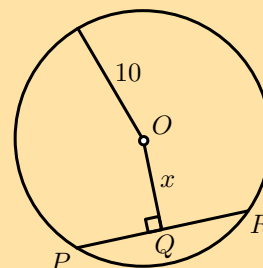
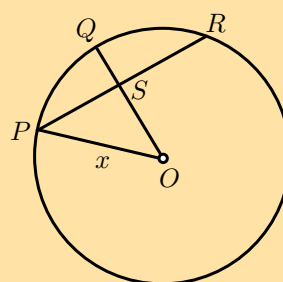
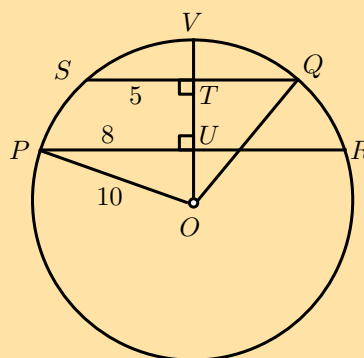
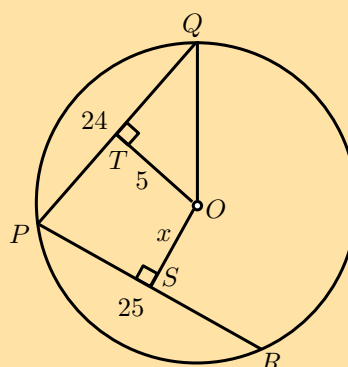
In $\triangle OQP$:

$$\begin{aligned} PQ &= 4 && (\perp \text{ from centre bisects chord}) \\ OP^2 &= OQ^2 + QP^2 && (\text{Pythagoras}) \\ 5^2 &= x^2 + 4^2 \\ \therefore x^2 &= 25 - 16 \\ x^2 &= 9 \\ x &= 3 \end{aligned}$$

Step 3: Write the final answer

$x = 3$ units.

Exercise 8 – 1: Perpendicular line from center bisects chord

1. In the circle with centre O, $OQ \perp PR$, $OQ = 4$ units and $PR = 10$. Determine x.

2. In the circle with centre O and radius $= 10$ units, $OQ \perp PR$ and $PR = 8$. Determine x.

3. In the circle with centre O, $OQ \perp PR$, $PR = 12$ units and $SQ = 2$ units. Determine x.

4. In the circle with centre O, $OT \perp SQ$, $OT \perp PR$, $OP = 10$ units, $ST = 5$ units and $PU = 8$ units. Determine TU.

5. In the circle with centre O, $OT \perp QP$, $OS \perp PR$, $OT = 5$ units, $PQ = 24$ units and $PR = 25$ units. Determine $OS = x$.

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1. 2367 2. 2368 3. 2369 4. 236B 5. 236C

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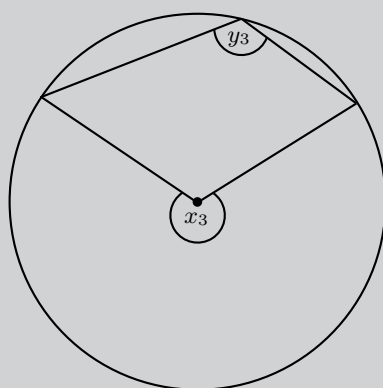
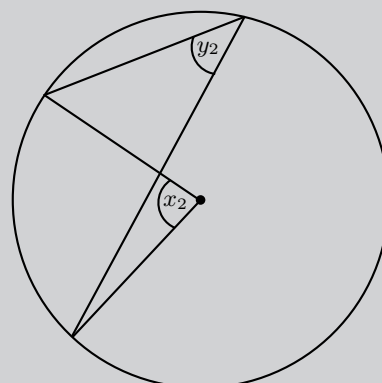
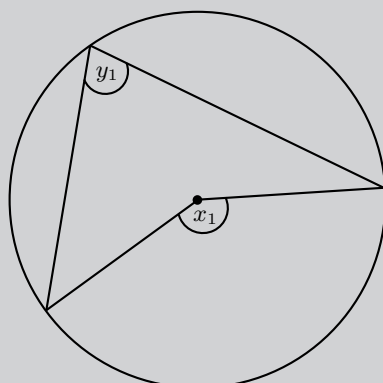
Investigation: Angles subtended by an arc at the centre and the circumference of a circle

1. Measure angles x and y in each of the following graphs:

2. Complete the table:

x	y

3. Use your results to make a conjecture about the relationship between angles subtended by an arc at the centre of a circle and angles at the circumference of a circle.

4. Now draw three of your own similar diagrams and measure the angles to check your conjecture.

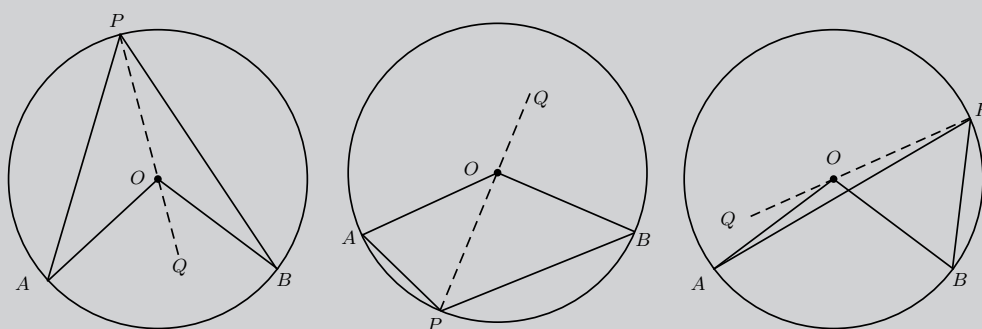
Theorem: Angle at the centre of a circle is twice the size of the angle at the circumference

STATEMENT

If an arc subtends an angle at the centre of a circle and at the circumference, then the angle at the centre is twice the size of the angle at the circumference.

(Reason: $\angle$ at centre $= 2\angle$ at circum.)

Given:

Circle with centre O, arc AB subtending $A\hat{O}B$ at the centre of the circle, and $A\hat{P}B$ at the circumference.

Required to prove:

$A\hat{O}B = 2A\hat{P}B$

PROOF

Draw PO extended to Q and let $A\hat{O}Q = \hat{O}_1$ and $B\hat{O}Q = \hat{O}_2$.

$$\begin{aligned} \hat{O}_1 &= A\hat{P}O + P\hat{A}O && \text{(ext. } \angle\triangle = \text{sum int. opp. } \angle\text{s)} \\ \text{and } A\hat{P}O &= P\hat{A}O && \text{(equal radii, isosceles } \triangle APO\text{)} \\ \therefore \hat{O}_1 &= A\hat{P}O + A\hat{P}O && \\ \hat{O}_1 &= 2A\hat{P}O && \end{aligned}$$

Similarly, we can also show that $\hat{O}_2 = 2B\hat{P}O$.

For the first two diagrams shown above we have that:

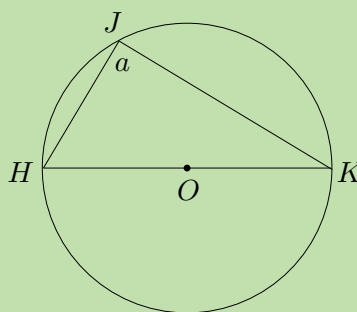
$$\begin{aligned} A\hat{O}B &= \hat{O}_1 + \hat{O}_2 \\ &= 2A\hat{P}O + 2B\hat{P}O \\ &= 2(A\hat{P}O + B\hat{P}O) \\ \therefore A\hat{O}B &= 2(A\hat{P}B) \end{aligned}$$

And for the last diagram:

$$\begin{aligned} A\hat{O}B &= \hat{O}_2 - \hat{O}_1 \\ &= 2B\hat{P}O - 2A\hat{P}O \\ &= 2(B\hat{P}O - A\hat{P}O) \\ \therefore A\hat{O}B &= 2(A\hat{P}B) \end{aligned}$$

Worked example 2: Angle at the centre of circle is twice angle at circumference

QUESTION

Given HK, the diameter of the circle passing through centre O.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles and sides on the diagram

Step 2: Solve for a

In $\triangle HJK$:

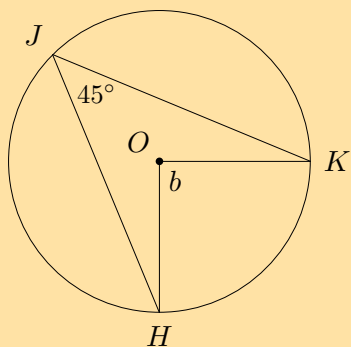
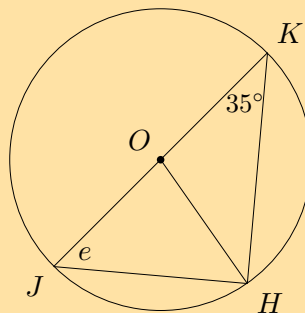
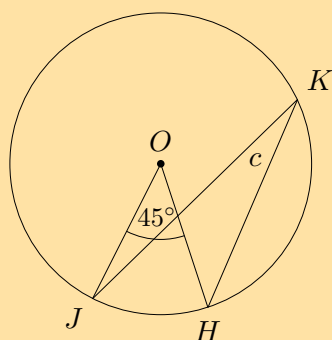
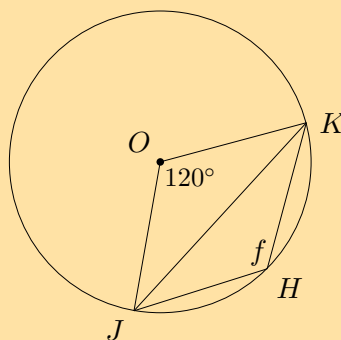
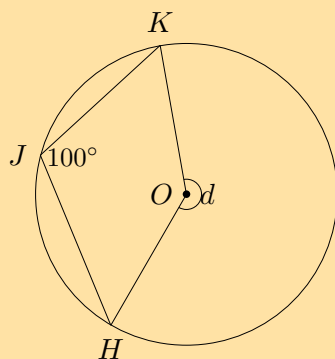
$$\begin{aligned} H\hat{O}K &= 180^\circ && (\angle \text{ on str. line}) \\ &= 2a && (\angle \text{ at centre } = 2\angle \text{ at circum.}) \\ \therefore 2a &= 180^\circ \\ a &= \frac{180^\circ}{2} \\ &= 90^\circ \end{aligned}$$

Step 3: Conclusion

The diameter of a circle subtends a right angle at the circumference (angles in a semi-circle).

Exercise 8 – 2: Angle at the centre of circle is twice angle at circumference

Given O is the centre of the circle, determine the unknown angle in each of the following diagrams:

1.

2.

3.

4.

5.

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1. 236D 2. 236F 3. 236G 4. 236H 5. 236J

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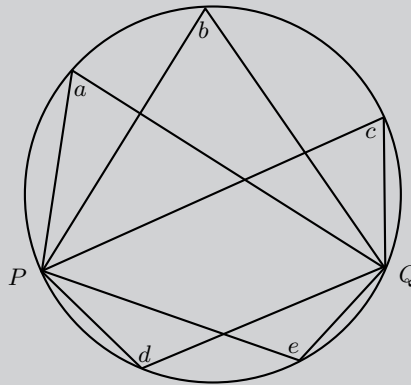
Investigation: Subtended angles in the same segment of a circle

1. Measure angles a, b, c, d and e in the diagram below:

2. Choose any two points on the circumference of the circle and label them A and B.
3. Draw AP and BP, and measure $A\hat{P}B$.
4. Draw AQ and BQ, and measure $A\hat{Q}B$.
5. What do you observe? Make a conjecture about these types of angles.

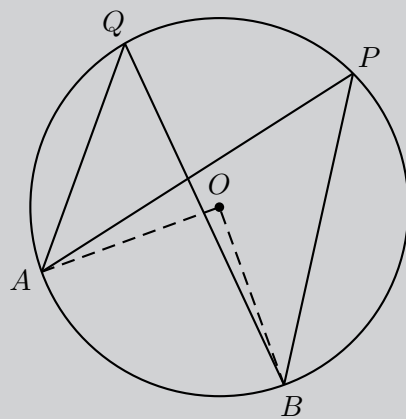
Theorem: Subtended angles in the same segment of a circle are equal

STATEMENT

If the angles subtended by a chord of the circle are on the same side of the chord, then the angles are equal.

(Reason: $\angle$s in same seg.)

Given:

Circle with centre O, and points P and Q on the circumference of the circle. Arc AB subtends $A\hat{P}B$ and $A\hat{Q}B$ in the same segment of the circle.

Required to prove:

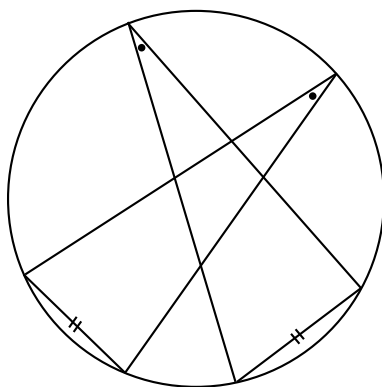
$A\hat{P}B = A\hat{Q}B$

PROOF

$$\begin{aligned} A\hat{O}B &= 2A\hat{P}B && (\angle \text{ at centre } = 2\angle \text{ at circum.}) \\ A\hat{O}B &= 2A\hat{Q}B && (\angle \text{ at centre } = 2\angle \text{ at circum.}) \\ \therefore 2A\hat{P}B &= 2A\hat{Q}B \\ A\hat{P}B &= A\hat{Q}B \end{aligned}$$

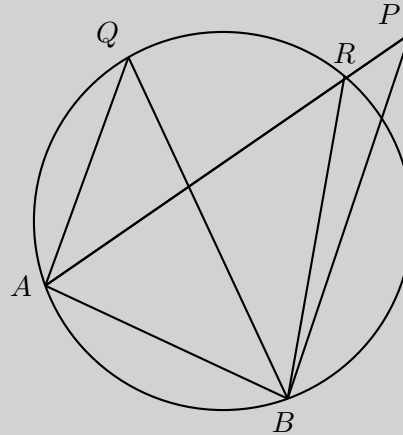
Equal arcs subtend equal angles

From the theorem above we can deduce that if angles at the circumference of a circle are subtended by arcs of equal length, then the angles are equal. In the figure below, notice that if we were to move the two chords with equal length closer to each other, until they overlap, we would have the same situation as with the theorem above. This shows that the angles subtended by arcs of equal length are also equal.

(PROOF NOT FOR EXAMS) Converse: Concyclic points

STATEMENT

If a line segment subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (lie on a circle).

Given:

Line segment AB subtending equal angles at points P and Q on the same side of the line segment AB.

Required to prove:

A, B, P and Q lie on a circle.

PROOF

Proof by contradiction:

Points on the circumference of a circle: we know that there are only two possible options regarding a given point — it either lies on circumference or it does not.

We will assume that point P does not lie on the circumference.

We draw a circle that cuts AP at R and passes through A, B and Q.

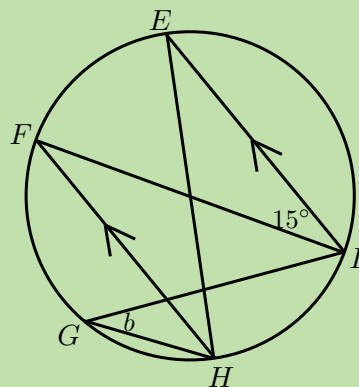
$$
\begin{aligned}
A\hat{Q}B &= A\hat{R}B && (\angle\text{s in same seg.})\\
\text{but } A\hat{Q}B &= A\hat{P}B && (\text{given})\\
\therefore A\hat{R}B &= A\hat{P}B &&\\
\text{but } A\hat{R}B &= A\hat{P}B + R\hat{B}P && (\text{ext. } \angle\triangle = \text{sum int. opp.})\\
\therefore R\hat{B}P &= 0^\circ &&
\end{aligned}
$$

Therefore the assumption that the circle does not pass through P must be false.

We can conclude that A, B, Q and P lie on a circle (A, B, Q and P are concyclic).

Worked example 3: Concyclic points

QUESTION

Given $FH \parallel EI$ and $E\hat{I}F = 15°$, determine the value of b.

SOLUTION

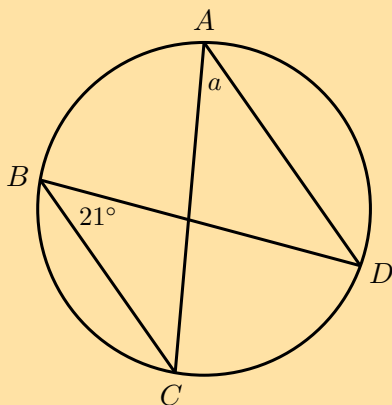
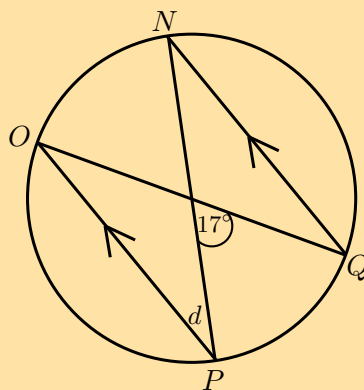
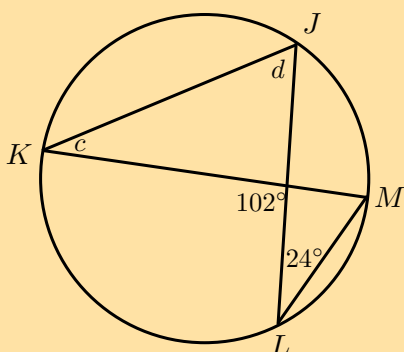
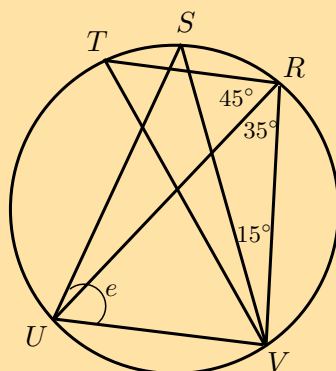
Step 1: Use theorems and the given information to find all equal angles on the diagram

Step 2: Solve for b

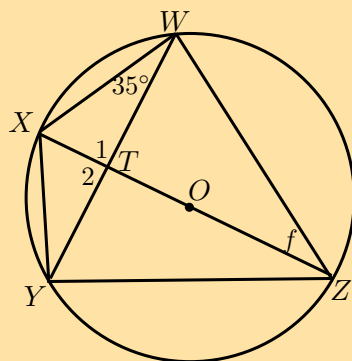
$$\begin{aligned} H\hat{F}I &= 15° && \text{(alt. } \angle, FH \parallel EI) \\ \text{and } b &= H\hat{F}I && (\angle\text{s in same seg.}) \\ \therefore b &= 15° \end{aligned}$$

Exercise 8 – 3: Subtended angles in the same segment

1. Find the values of the unknown angles.

a)

c)

b)

2.

a) Given $T\hat{V}S = S\hat{V}R$, determine the value of e.

b) Is TV a diameter of the circle? Explain your answer.

3.

Given circle with centre O, $WT = TY$ and $X\hat{W}T = 35°$. Determine f.

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1a. 236K 1b. 236M 1c. 236N 2. 236P 3. 236Q

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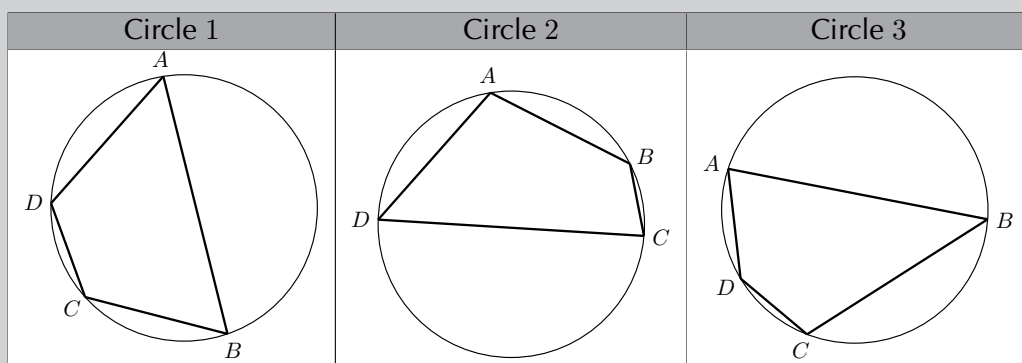
Cyclic quadrilaterals

Cyclic quadrilaterals are quadrilaterals with all four vertices lying on the circumference of a circle (concyclic).

Investigation: Cyclic quadrilaterals

Consider the diagrams given below:

1. Complete the following:

   $ABCD$ is a cyclic quadrilateral because ......

2. Complete the table:

	Circle 1	Circle 2	Circle 3
$\hat{A} =$			
$\hat{B} =$			
$\hat{C} =$			
$\hat{D} =$			
$\hat{A} + \hat{C} =$			
$\hat{B} + \hat{D} =$			

3. Use your results to make a conjecture about the relationship between angles of cyclic quadrilaterals.

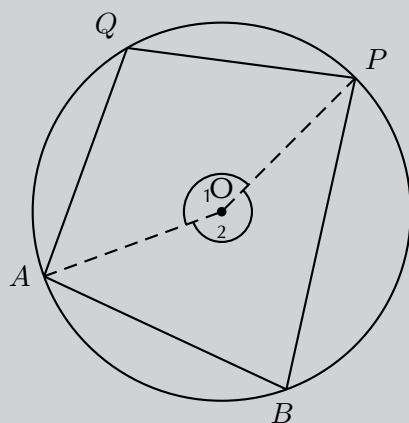
Theorem: Opposite angles of a cyclic quadrilateral

STATEMENT

The opposite angles of a cyclic quadrilateral are supplementary.

(Reason: opp. $\angle$s cyclic quad.)

Given:

Circle with centre O with points A, B, P and Q on the circumference such that $ABPQ$ is a cyclic quadrilateral.

Required to prove:

$A\hat{B}P + A\hat{Q}P = 180°$ and $Q\hat{A}B + Q\hat{P}B = 180°$

PROOF

Draw AO and OP. Label $\hat{O}_1$ and $\hat{O}_2$.

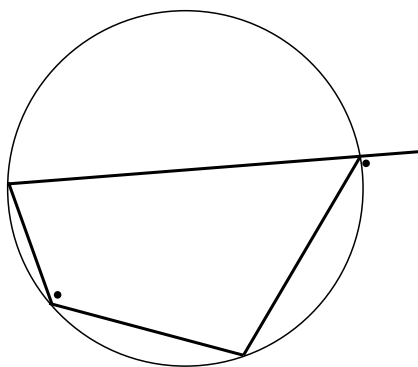
$$\hat{O}_1 = 2A\hat{B}P \quad (\angle \text{ at centre} = 2\angle \text{ at circum.})$$
$$\hat{O}_2 = 2A\hat{Q}P \quad (\angle \text{ at centre} = 2\angle \text{ at circum.})$$
$$\text{and } \hat{O}_1 + \hat{O}_2 = 360° \quad (\angle\text{s around a point})$$
$$\therefore 2A\hat{B}P + 2A\hat{Q}P = 360°$$
$$A\hat{B}P + A\hat{Q}P = 180°$$

Similarly, we can show that $Q\hat{A}B + Q\hat{P}B = 180°$.

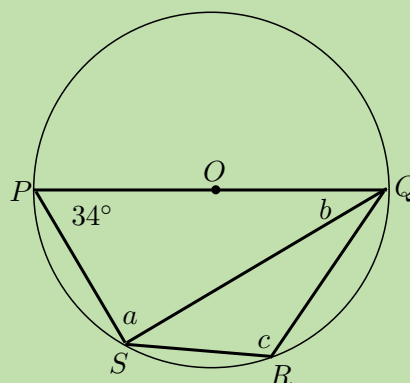
Converse: interior opposite angles of a quadrilateral

If the interior opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Exterior angle of a cyclic quadrilateral

If a quadrilateral is cyclic, then the exterior angle is equal to the interior opposite angle.

Worked example 4: Opposite angles of a cyclic quadrilateral

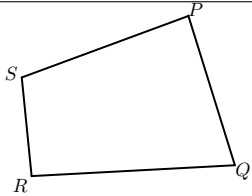
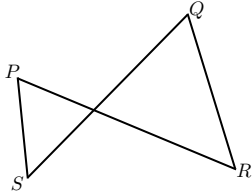
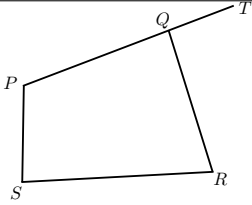
QUESTION

Given the circle with centre O and cyclic quadrilateral $PQRS$. SQ is drawn and $S\hat{P}Q = 34°$. Determine the values of a, b and c.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles on the diagram

Step 2: Solve for b

$$\begin{aligned} S\hat{P}Q + c &= 180° && \text{(opp. } \angle\text{s cyclic quad supp.)} \\ \therefore c &= 180° - 34° \\ &= 146° \end{aligned}$$

$$a = 90° \qquad (\angle \text{ in semi circle})$$

In $\triangle PSQ$:

$$\begin{aligned} a + b + 34° &= 180° && (\angle \text{ sum of } \triangle) \\ \therefore b &= 180° - 90° - 34° \\ &= 56° \end{aligned}$$

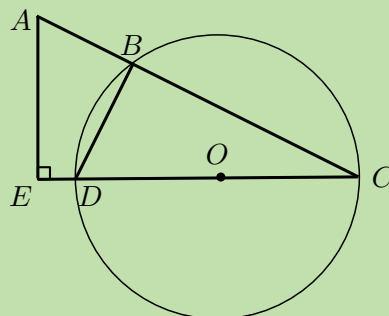
Methods for proving a quadrilateral is cyclic

There are three ways to prove that a quadrilateral is a cyclic quadrilateral:

	Method of proof	Reason
S P Q R	If $\hat{P} + \hat{R} = 180°$ or $\hat{S} + \hat{Q} = 180°$, then $PQRS$ is a cyclic quad.	opp. int. angles suppl.
P Q R S	If $\hat{P} = \hat{Q}$ or $\hat{S} = \hat{R}$, then $PQRS$ is a cyclic quad.	angles in the same seg.
P Q T R S	If $T\hat{Q}R = \hat{S}$, then $PQRS$ is a cyclic quad.	ext. angle equal to int. opp. angle

Worked example 5: Proving a quadrilateral is a cyclic quadrilateral

QUESTION

Prove that $ABDE$ is a cyclic quadrilateral.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles on the diagram

Step 2: Prove that $ABDE$ is a cyclic quadrilateral

$D\hat{B}C = 90°$ ($\angle$ in semi circle)

and $\hat{E} = 90°$ (given)

$\therefore D\hat{B}C = \hat{E}$

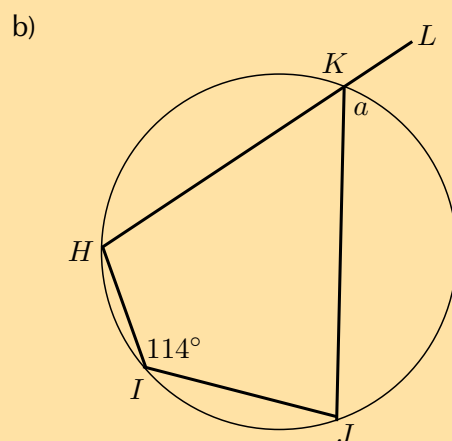
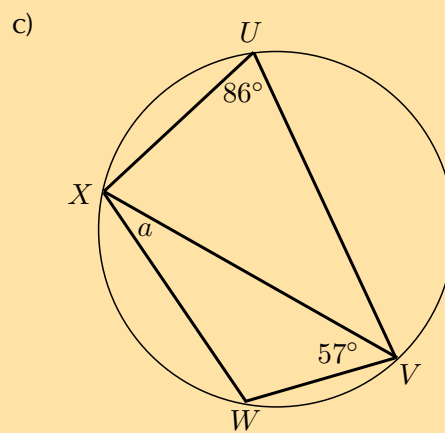
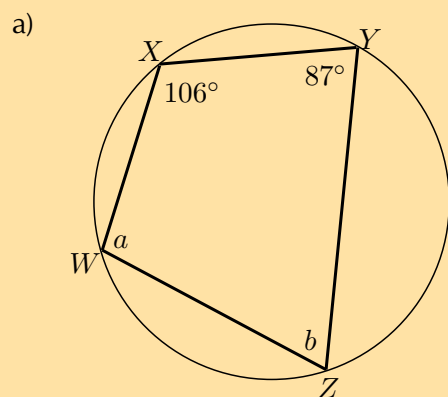
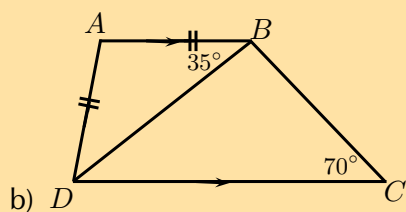
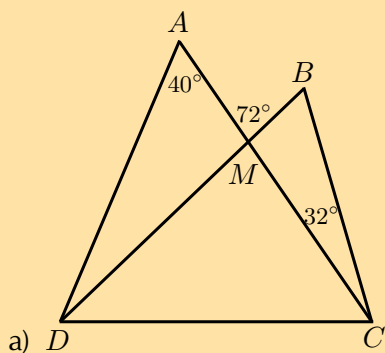
$\therefore ABDE$ is a cyclic quadrilateral (ext. $\angle$ equals int. opp. $\angle$)

Exercise 8 – 4: Cyclic quadrilaterals

1. Find the values of the unknown angles.

a)

c)

b)

2. Prove that $ABCD$ is a cyclic quadrilateral:

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 236R 1b. 236S 1c. 236T 2a. 236V 2b. 236W

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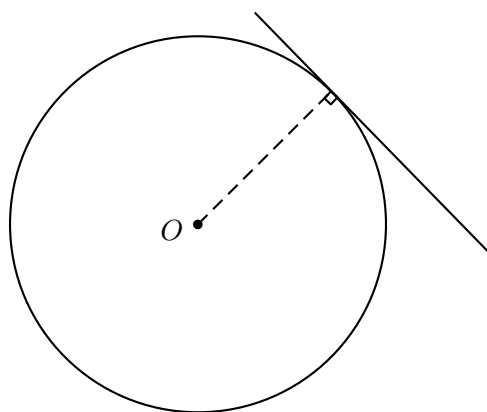
Tangent line to a circle

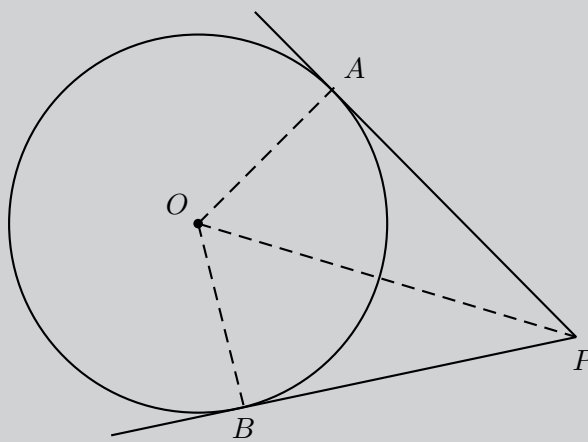
A tangent is a line that touches the circumference of a circle at only one place. The radius of a circle is perpendicular to the tangent at the point of contact.

Theorem: Two tangents drawn from the same point outside a circle

STATEMENT

If two tangents are drawn from the same point outside a circle, then they are equal in length.

(Reason: tangents from same point equal)

Given:

Circle with centre O and tangents PA and PB, where A and B are the respective points of contact for the two lines.

Required to prove:

$AP = BP$

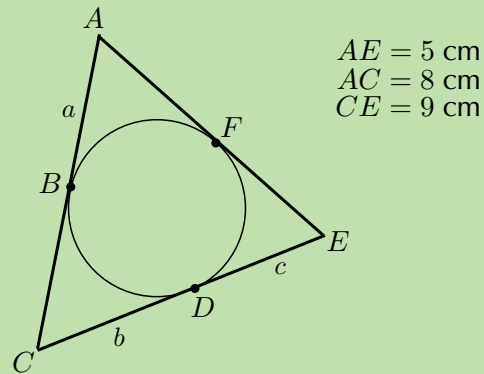
PROOF

In $\triangle AOP$ and $\triangle BOP$,

$$
\begin{aligned}
O\hat{A}P = O\hat{B}P &= 90^\circ && \text{(tangent } \perp \text{ radius)} \\
AO &= BO && \text{(equal radii)} \\
OP &= OP && \text{(common side)} \\
\therefore \triangle AOP &\equiv \triangle BOP && \text{(RHS)} \\
\therefore AP &= BP
\end{aligned}
$$

Worked example 6: Tangents from the same point outside a circle

QUESTION

In the diagram below $AE = 5$ cm, $AC = 8$ cm and $CE = 9$ cm. Determine the values of a, b and c.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles on the diagram

Step 2: Solve for a, b and c

$$
\begin{aligned}
AB = AF &= a && \text{(tangents from } A\text{)}\\
EF = ED &= c && \text{(tangents from } E\text{)}\\
CB = CD &= b && \text{(tangents from } C\text{)}\\
\therefore AE = a + c &= 5 \\
\text{and } AC = a + b &= 8 \\
\text{and } CE = b + c &= 9
\end{aligned}
$$

Step 3: Solve for the unknown variables using simultaneous equations

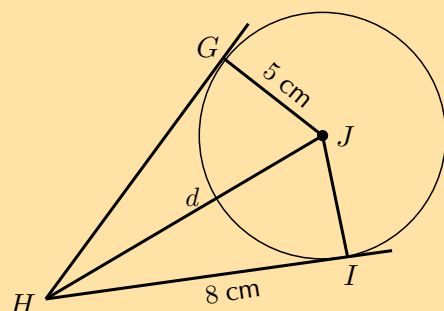
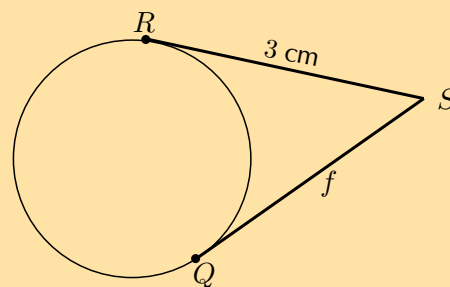
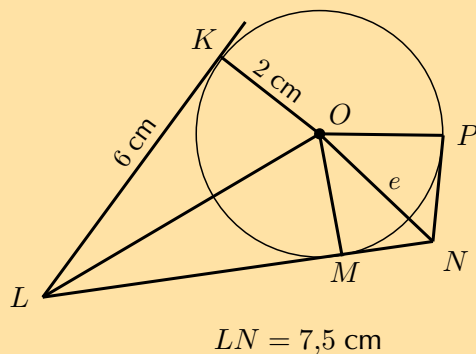
$$
\begin{aligned}
a + c &= 5 && \ldots(1)\\
a + b &= 8 && \ldots(2)\\
b + c &= 9 && \ldots(3)
\end{aligned}
$$

Subtract equation (1) from equation (2) and then substitute into equation (3):

$$
\begin{aligned}
(2) - (1) \quad b - c &= 8 - 5 \\
&= 3 \\
\therefore b &= c + 3 \\
\text{Substitute into (3)} \quad c + 3 + c &= 9 \\
2c &= 6 \\
c &= 3 \\
\therefore a &= 2 \\
\text{and } b &= 6
\end{aligned}
$$

Exercise 8 – 5: Tangents to a circle

Find the values of the unknown lengths.

1.

3.

2.

$LN = 7{,}5$ cm

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1. 236X 2. 236Y 3. 236Z

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Investigation: Tangent-chord theorem

Consider the diagrams given below:

Diagram 1

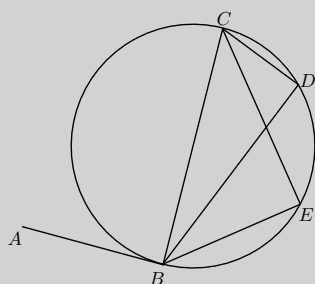



Diagram 2

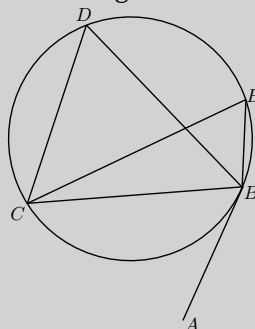
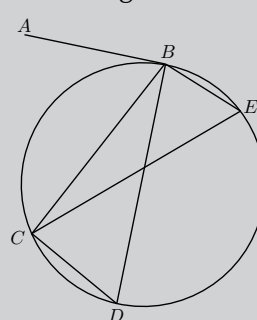



Diagram 3

1. Measure the following angles with a protractor and complete the table:

	Diagram 1	Diagram 2	Diagram 3
$A\hat{B}C =$			
$\hat{D} =$			
$\hat{E} =$			

2. Use your results to complete the following: the angle between a tangent to a circle and a chord is . . . . . . to the angle in the alternate segment.

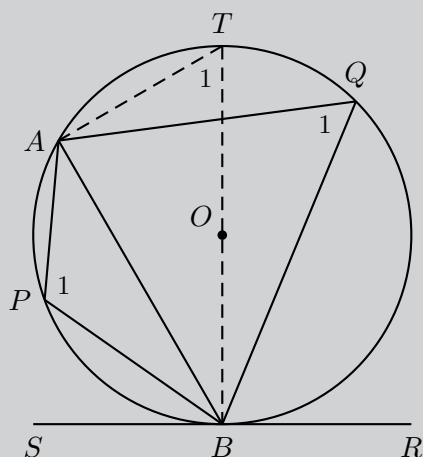
Theorem: Tangent-chord theorem

STATEMENT

The angle between a tangent to a circle and a chord drawn at the point of contact, is equal to the angle which the chord subtends in the alternate segment.

(Reason: tan. chord theorem)

Given:

Circle with centre O and tangent SR touching the circle at B. Chord AB subtends $\hat{P}_1$ and $\hat{Q}_1$.

Required to prove:

1. $A\hat{B}R = A\hat{P}B$
2. $A\hat{B}S = A\hat{Q}B$

PROOF

Draw diameter BT and join T to A.

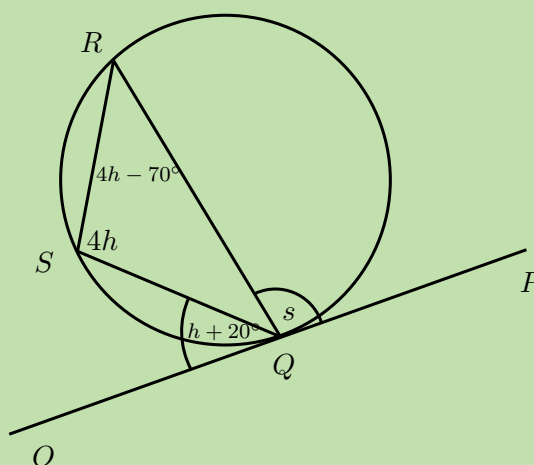
Let $A\hat{T}B = T_1$.

$$A\hat{B}S + A\hat{B}T = 90^\circ \quad \text{(tangent } \perp \text{ radius)}$$
$$B\hat{A}T = 90^\circ \quad (\angle \text{ in semi circle})$$
$$\therefore A\hat{B}T + T_1 = 90^\circ \quad (\angle \text{ sum of } \triangle BAT)$$
$$\therefore A\hat{B}S = T_1$$
$$\text{but } Q_1 = T_1 \quad (\angle\text{s in same segment})$$
$$\therefore Q_1 = A\hat{B}S$$

$$A\hat{B}S + A\hat{B}R = 180^\circ \quad (\angle\text{s on str. line})$$
$$\hat{Q}_1 + \hat{P}_1 = 180^\circ \quad (\text{opp. } \angle\text{s cyclic quad. supp.})$$
$$\therefore A\hat{B}S + A\hat{B}R = Q_1 + P_1$$
$$\text{and } A\hat{B}S = Q_1$$
$$\therefore A\hat{B}R = P_1$$

Worked example 7: Tangent-chord theorem

QUESTION

Determine the values of h and s.

SOLUTION

Step 1: Use theorems and the given information to find all equal angles on the diagram

Step 2: Solve for h

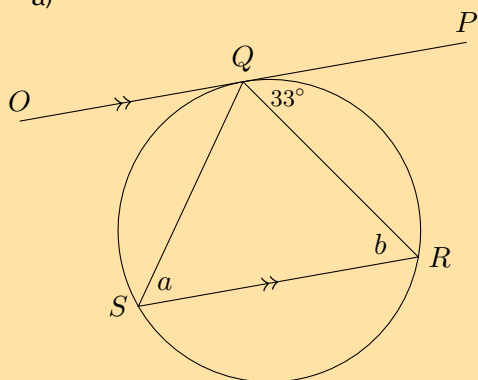
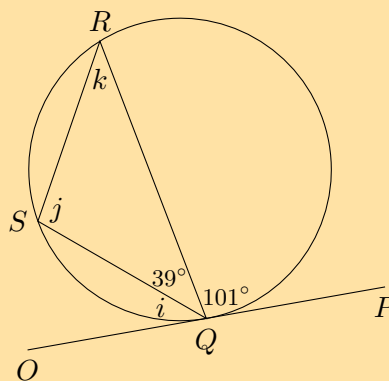
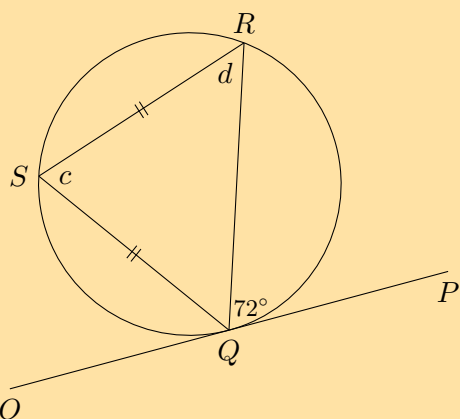
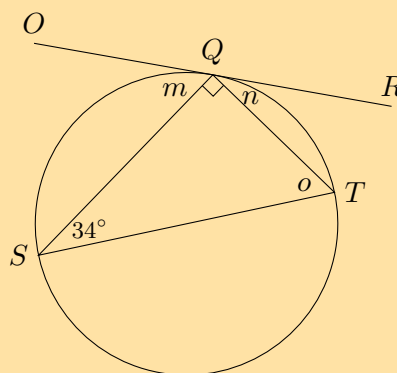
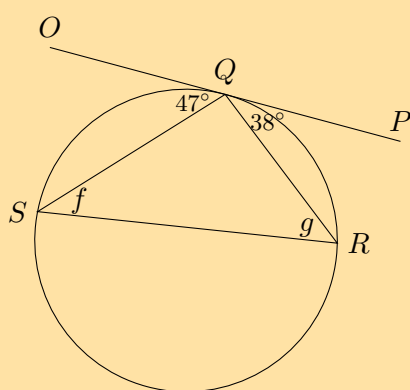
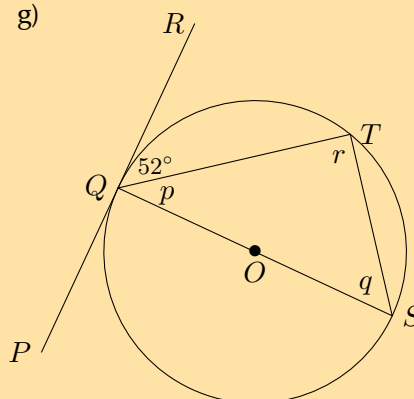
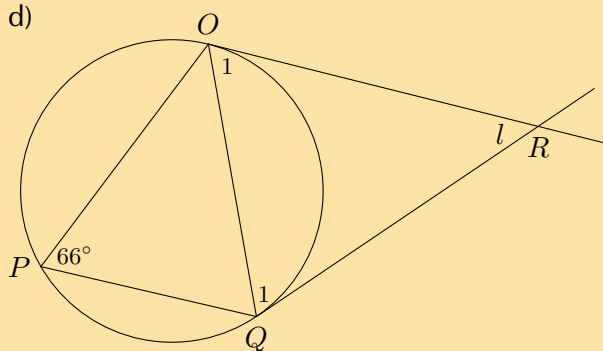
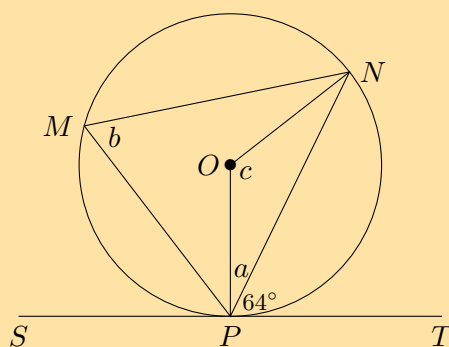
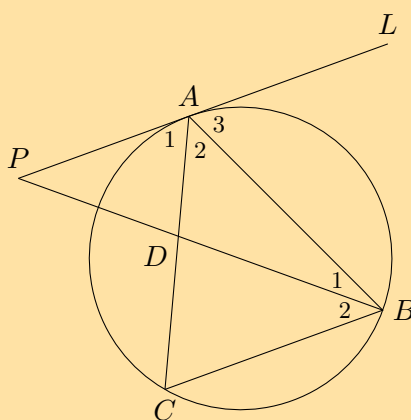
$$\begin{aligned} O\hat{Q}S &= S\hat{R}Q && \text{(tangent chord theorem)} \\ h + 20° &= 4h - 70° \\ 90° &= 3h \\ \therefore h &= 30° \end{aligned}$$

Step 3: Solve for s

$$\begin{aligned} P\hat{Q}R &= Q\hat{S}R && \text{(tangent chord theorem)} \\ s &= 4h \\ &= 4(30°) \\ &= 120° \end{aligned}$$

Exercise 8 – 6: Tangent-chord theorem

1. Find the values of the unknown letters, stating reasons.

a)

e)

b)

f)

c)

g)

d)

2. O is the centre of the circle and SPT is a tangent, with $OP \perp ST$. Determine a, b and c, giving reasons.

3.

Given $AB = AC$, $AP \parallel BC$ and $\hat{A}_2 = \hat{B}_2$. Prove:

a) PAL is a tangent to the circle ABC.

b) AB is a tangent to the circle ADP.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1a. 2372 1b. 2373 1c. 2374 1d. 2375 1e. 2376 1f. 2377
1g. 2378 2. 2379 3. 237B

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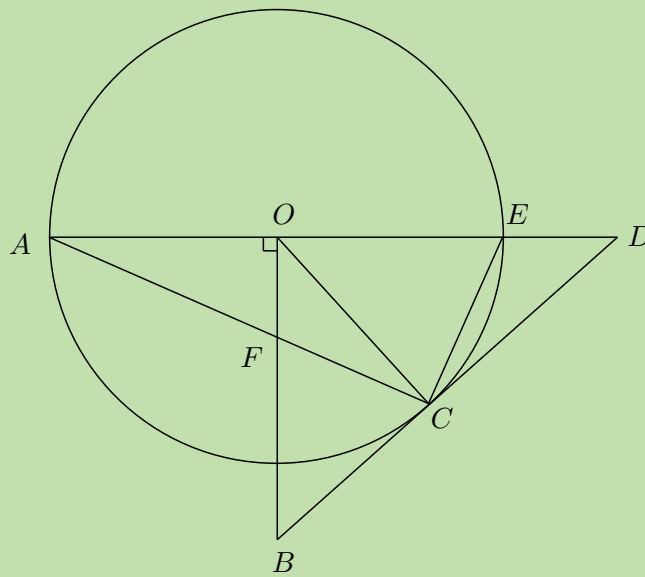
Converse: tangent-chord theorem

If a line drawn through the end point of a chord forms an angle equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

(Reason: $\angle$ between line and chord $= \angle$ in alt. seg.)

Worked example 8: Applying the theorems

QUESTION

BD is a tangent to the circle with centre O, with $BO \perp AD$.

Prove that:

1. $CFOE$ is a cyclic quadrilateral
2. $FB = BC$
3. $\angle A\hat{O}C = 2B\hat{F}C$
4. Will DC be a tangent to the circle passing through C, F, O and E? Motivate your answer.

SOLUTION

Step 1: Prove $CFOE$ is a cyclic quadrilateral by showing opposite angles are supplementary

$$\begin{array}{rl} BO \perp OD & \text{(given)} \\ \therefore F\hat{O}E = 90^\circ & \\ F\hat{C}E = 90^\circ & (\angle \text{ in semi circle}) \\ \therefore CFOE \text{ is a cyclic quad.} & (\text{opp. } \angle\text{s suppl.}) \end{array}$$

Step 2: Prove BFC is an isosceles triangle

To show that $FB = BC$ we first prove $\triangle BFC$ is an isosceles triangle by showing that $B\hat{F}C = B\hat{C}F$.

$$\begin{array}{rl} B\hat{C}F = C\hat{E}O & \text{(tangent-chord)} \\ C\hat{E}O = B\hat{F}C & (\text{ext. } \angle \text{ cyclic quad. } CFOE) \\ \therefore B\hat{F}C = B\hat{C}F & \\ \therefore FB = BC & (\triangle BFC \text{ isosceles}) \end{array}$$

Step 3: Prove $A\hat{O}C = 2B\hat{F}C$

$$\begin{aligned} A\hat{O}C &= 2A\hat{E}C && (\angle \text{ at centre } = 2\angle \text{ at circum.}) \\ \text{and } A\hat{E}C &= B\hat{F}C && (\text{ext. } \angle \text{ cyclic quad. } CFOE) \\ \therefore A\hat{O}C &= 2B\hat{F}C \end{aligned}$$

Step 4: Determine if DC is a tangent to the circle through C, F, O and E

Proof by contradiction.

Let us assume that DC is a tangent to the circle passing through the points C, F, O and E:

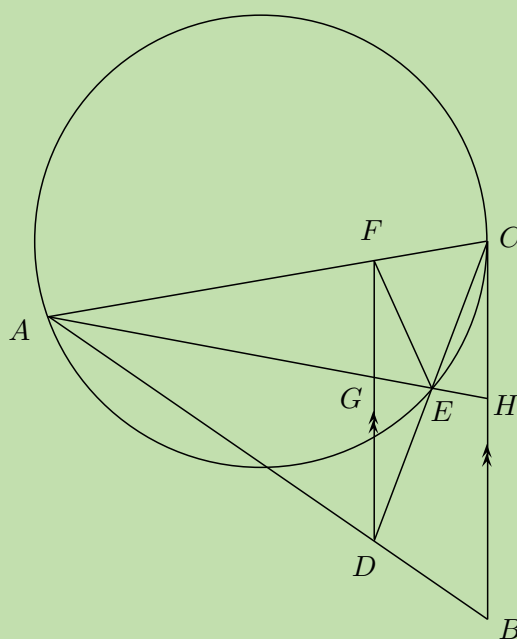
$$\therefore D\hat{C}E = C\hat{O}E \quad (\text{tangent-chord})$$

And using the circle with centre O and tangent BD we have that:

$$\begin{aligned} D\hat{C}E &= C\hat{A}E && (\text{tangent-chord}) \\ \text{but } C\hat{A}E &= \tfrac{1}{2}C\hat{O}E && (\angle \text{ at centre } = 2\angle \text{ at circum.}) \\ \therefore D\hat{C}E &\neq C\hat{O}E \end{aligned}$$

Therefore our assumption is not correct and we can conclude that DC is not a tangent to the circle passing through the points C, F, O and E.

Worked example 9: Applying the theorems

QUESTION

FD is drawn parallel to the tangent CB

Prove that:

1. $FADE$ is a cyclic quadrilateral
2. $F\hat{E}A = \hat{B}$

SOLUTION

Step 1: Prove $FADE$ is a cyclic quadrilateral using angles in the same segment

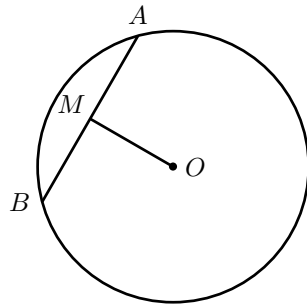
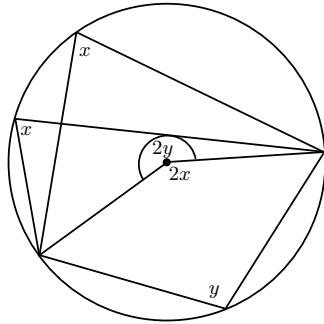
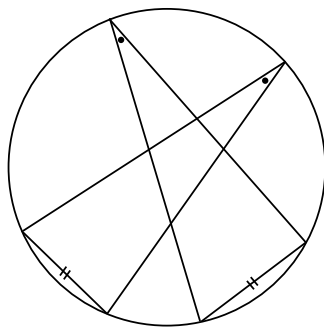
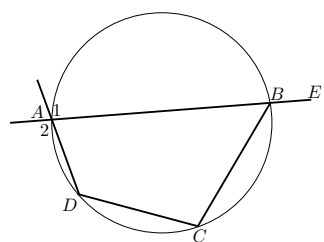
$$
\begin{aligned}
F\hat{D}C &= D\hat{C}B && \text{(alt. } \angle\text{s } FD \parallel CB) \\
\text{and } D\hat{C}B &= C\hat{A}E && \text{(tangent-chord)} \\
\therefore F\hat{D}C &= C\hat{A}E && \\
\therefore FADE & \text{ is a cyclic quad.} && (\angle\text{s in same seg.})
\end{aligned}
$$

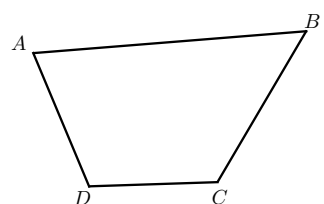
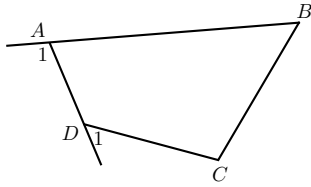
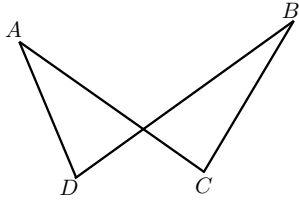
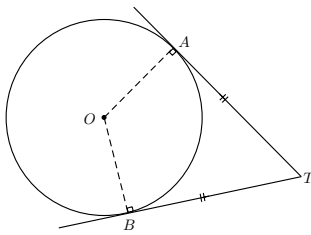
Step 2: Prove $F\hat{E}A = \hat{B}$

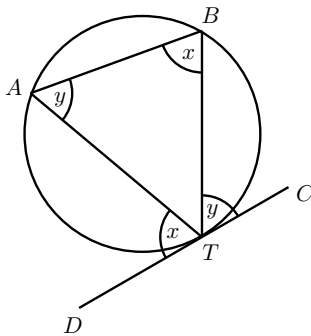
$$
\begin{aligned}
F\hat{D}A &= \hat{B} && \text{(corresp. } \angle\text{s } FD \parallel CB) \\
\text{and } F\hat{E}A &= F\hat{D}A && (\angle\text{s same seg. cyclic quad. } FADE) \\
\therefore F\hat{E}A &= \hat{B} &&
\end{aligned}
$$

8.3 Summary EMBJC

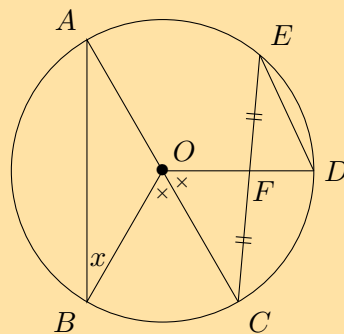
See presentation: 237C at www.everythingmaths.co.za

- **Arc** An arc is a portion of the circumference of a circle.
- **Chord** - a straight line joining the ends of an arc.
- **Circumference** - perimeter or boundary line of a circle.
- **Radius** (r) - any straight line from the centre of the circle to a point on the circumference.
- **Diameter** - a special chord that passes through the centre of the circle. A diameter is the length of a straight line segment from one point on the circumference to another point on the circumference, that passes through the centre of the circle.
- **Segment** A segment is a part of the circle that is cut off by a chord. A chord divides a circle into two segments.
- **Tangent** - a straight line that makes contact with a circle at only one point on the circumference.
- A tangent line is perpendicular to the radius, drawn at the point of contact with the circle.

- If O is the centre and $OM \perp AB$, then $AM = MB$.
- If O is the centre and $AM = MB$, then $A\hat{M}O = B\hat{M}O = 90°$.
- If $AM = MB$ and $OM \perp AB$, then $\Rightarrow MO$ passes through centre O.

If an arc subtends an angle at the centre of a circle and at the circumference, then the angle at the centre is twice the size of the angle at the circumference.

Angles at the circumference subtended by the same arc (or arcs of equal length) are equal.

The four sides of a cyclic quadrilateral $ABCD$ are chords of the circle with centre O.

- $\hat{A} + \hat{C} = 180°$ (opp. $\angle$s supp.)
- $\hat{B} + \hat{D} = 180°$ (opp. $\angle$s supp.)
- $E\hat{B}C = \hat{D}$ (ext. $\angle$ cyclic quad.)
- $\hat{A}_1 = \hat{A}_2 = \hat{C}$ (vert. opp. $\angle$, ext. $\angle$ cyclic quad.)

Proving a quadrilateral is cyclic: If $\hat{A} + \hat{C} = 180°$ or $\hat{B} + \hat{D} = 180°$, then $ABCD$ is a cyclic quadrilateral.

If $\hat{A}_1 = \hat{C}$ or $\hat{D}_1 = \hat{B}$, then $ABCD$ is a cyclic quadrilateral.

If $\hat{A} = \hat{B}$ or $\hat{C} = \hat{D}$, then $ABCD$ is a cyclic quadrilateral.

If AT and BT are tangents to circle O, then

- $OA \perp AT$ (tangent $\perp$ radius)
- $OB \perp BT$ (tangent $\perp$ radius)
- $TA = TB$ (tangents from same point equal)

- If DC is a tangent, then $D\hat{T}A = T\hat{B}A$ and $C\hat{T}B = T\hat{A}B$
- If $D\hat{T}A = T\hat{B}A$ or $C\hat{T}B = T\hat{A}B$, then DC is a tangent touching at T

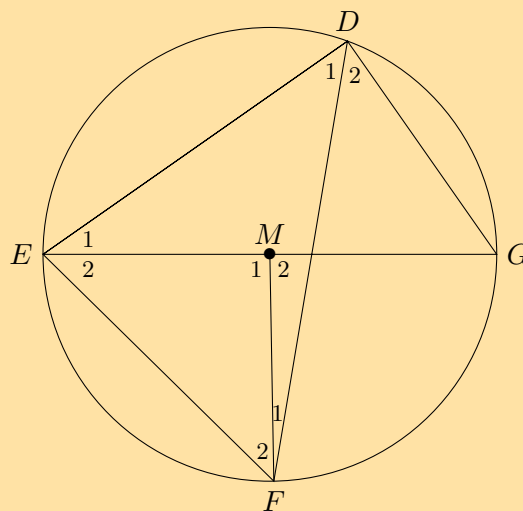
Exercise 8 – 7: End of chapter exercises

1.

AOC is a diameter of the circle with centre O. F is the mid-point of chord EC. $B\hat{O}C = C\hat{O}D$ and $\hat{B} = x$. Express the following angles in terms of x, stating reasons:

a) $\hat{A}$

b) $C\hat{O}D$

c) $\hat{D}$

2.

D, E, F and G are points on circle with centre M. $\hat{F}_1 = 7°$ and $\hat{D}_2 = 51°$. Determine the sizes of the following angles, stating reasons:

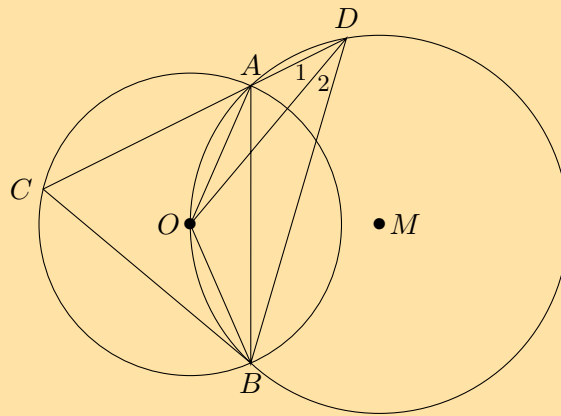
a) $\hat{M}_1$

b) $\hat{D}_1$

c) $\hat{F}_2$

d) $\hat{G}$

e) $\hat{E}_1$

3.

O is a point on the circle with centre M. O is also the centre of a second circle. DA cuts the smaller circle at C and $\hat{D}_1 = x$. Express the following angles in terms of x, stating reasons:

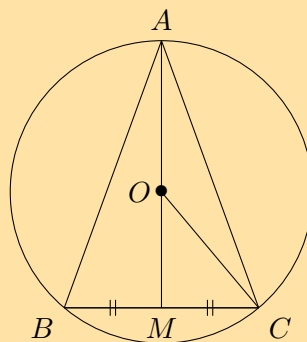
a) $\hat{D}_2$

b) $O\hat{A}B$

c) $O\hat{B}A$

d) $A\hat{O}B$

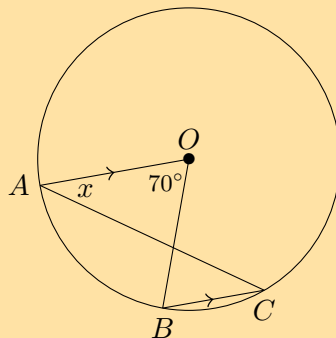
e) $\hat{C}$

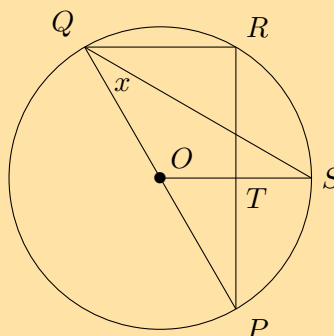
4.

O is the centre of the circle with radius 5 cm and chord $BC = 8$ cm. Calculate the lengths of:

a) OM

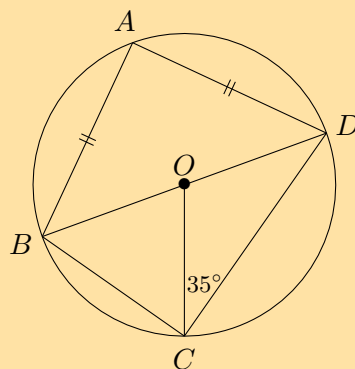
b) AM

c) AB

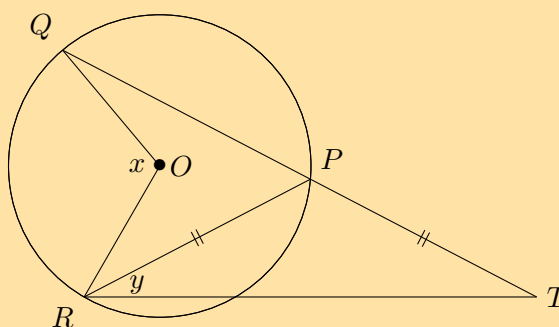
5.

$AO \parallel CB$ in circle with centre O. $A\hat{O}B = 70°$ and $O\hat{A}C = x$. Calculate the value of x, giving reasons.

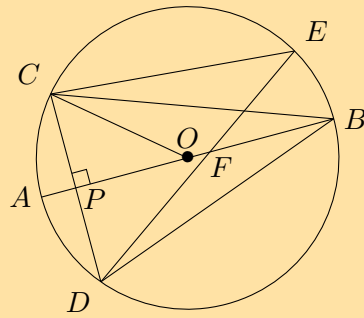
6.

PQ is a diameter of the circle with centre O. SQ bisects $P\hat{Q}R$ and $P\hat{Q}S = x$.

a) Write down two other angles that are also equal to x.
b) Calculate $P\hat{O}S$ in terms of x, giving reasons.
c) Prove that OS is a perpendicular bisector of PR.

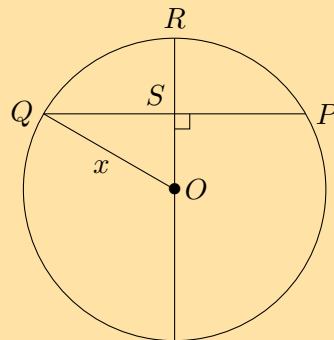
7.

$B\hat{O}D$ is a diameter of the circle with centre O. $AB = AD$ and $O\hat{C}D = 35°$. Calculate the value of the following angles, giving reasons:

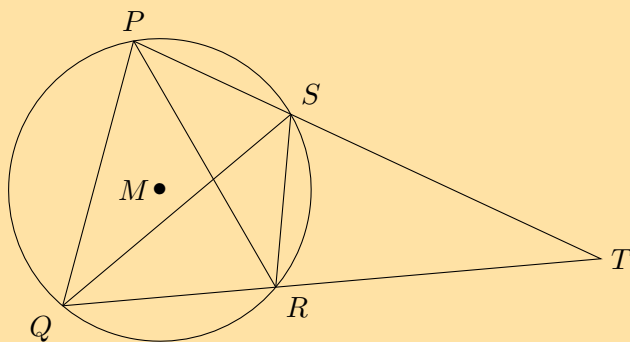
a) $O\hat{D}C$
b) $C\hat{O}D$
c) $C\hat{B}D$
d) $B\hat{A}D$
e) $A\hat{D}B$

8.

QP in the circle with centre O is protracted to T so that $PR = PT$. Express y in terms of x.

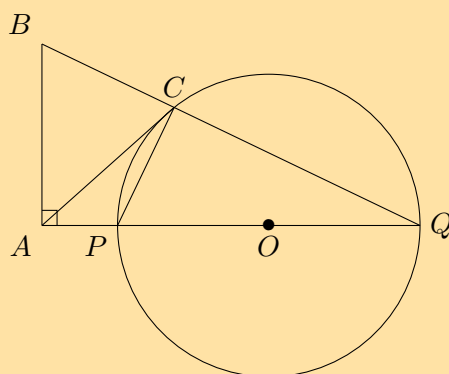
9.

O is the centre of the circle with diameter AB. $CD \perp AB$ at P and chord DE cuts AB at F. Prove that:

a) $C\hat{B}P = D\hat{P}B$
b) $C\hat{E}D = 2C\hat{B}A$
c) $A\hat{B}D = \frac{1}{2}C\hat{O}A$

10.

In the circle with centre O, $OR \perp QP$, $PQ = 30$ mm and $RS = 9$ mm. Determine the length of OQ.

11.

P, Q, R and S are points on the circle with centre M. PS and QR are extended and meet at T. $PQ = PR$ and $P\hat{Q}R = 70°$.

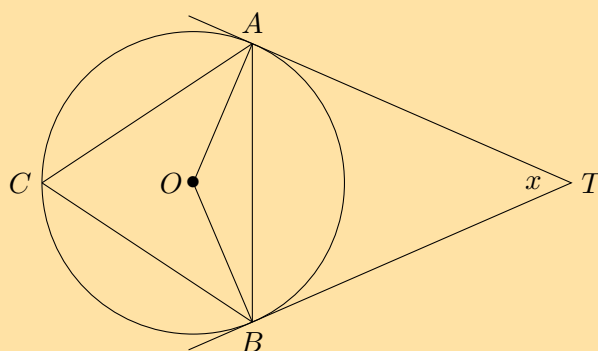
a) Determine, stating reasons, three more angles equal to $70°$.
b) If $Q\hat{P}S = 80°$, calculate $S\hat{R}T$, $S\hat{T}R$ and $P\hat{Q}S$.
c) Explain why PQ is a tangent to the circle QST at point Q.
d) Determine $P\hat{M}Q$.

12.

POQ is a diameter of the circle with centre O. QP is protruded to A and AC is a tangent to the circle. $BA \perp AQ$ and BCQ is a straight line. Prove:

a) $P\hat{C}Q = B\hat{A}P$

b) $BAPC$ is a cyclic quadrilateral

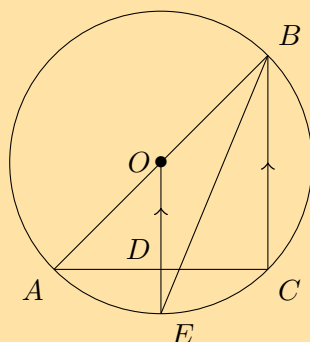
c) $AB = AC$

13.

TA and TB are tangents to the circle with centre O. C is a point on the circumference and $A\hat{T}B = x$. Express the following in terms of x, giving reasons:

a) $A\hat{B}T$

b) $O\hat{B}A$

c) $\hat{C}$

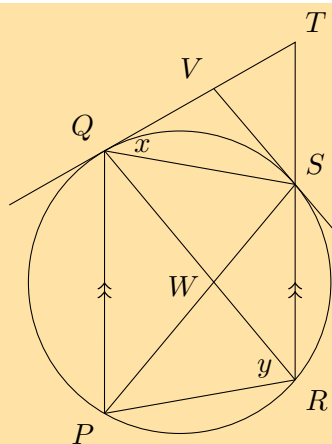
14.

AOB is a diameter of the circle $AECB$ with centre O. $OE \parallel BC$ and cuts AC at D.

a) Prove $AD = DC$

b) Show that $A\hat{B}C$ is bisected by EB

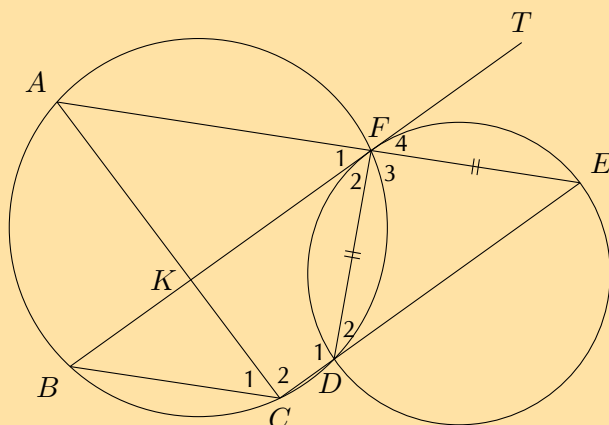
c) If $O\hat{E}B = x$, express $B\hat{A}C$ in terms of x

d) Calculate the radius of the circle if $AC = 10$ cm and $DE = 1$ cm

15.

PQ and RS are chords of the circle and $PQ \parallel RS$. The tangent to the circle at Q meets RS protruded at T. The tangent at S meets QT at V. QS and PR are drawn.

Let $T\hat{Q}S = x$ and $Q\hat{R}P = y$. Prove that:

a) $T\hat{V}S = 2Q\hat{R}S$
b) $QVSW$ is a cyclic quadrilateral
c) $Q\hat{P}S + \hat{T} = P\hat{R}T$
d) W is the centre of the circle

16.

The two circles shown intersect at points F and D. BFT is a tangent to the smaller circle at F. Straight line AFE is drawn such that $DF = EF$. CDE is a straight line and chord AC and BF cut at K. Prove that:

a) $BT \parallel CE$
b) $BCEF$ is a parallelogram
c) $AC = BF$

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1. 237D 2. 237F 3. 237G 4. 237H 5. 237J 6. 237K
7. 237M 8. 237N 9. 237P 10. 237Q 11. 237R 12. 237S
13. 237T 14. 237V 15. 237W 16. 237X

CHAPTER

Finance, growth and decay

9.1 Revision EMBJD

Simple interest is the interest calculated only on the initial amount invested, the principal amount. Compound interest is the interest earned on the principal amount and on its accumulated interest. This means that interest is being earned on interest. The accumulated amount is the final amount; the sum of the principal amount and the amount of interest earned.

Formula for simple interest:

$$A = P(1 + in)$$

Formula for compound interest:

$$A = P(1 + i)^n$$

where

$$\begin{aligned} A &= \text{accumulated amount} \\ P &= \text{principal amount} \\ i &= \text{interest rate written as a decimal} \\ n &= \text{time period in years} \end{aligned}$$

Worked example 1: Simple and compound interest

QUESTION

Sam wants to invest R 3450 for 5 years. Wise Bank offers a savings account which pays simple interest at a rate of 12,5% per annum, and Grand Bank offers a savings account paying compound interest at a rate of 10,4% per annum. Which bank account would give Sam the greatest accumulated balance at the end of the 5 year period?

SOLUTION

Step 1: Calculation using the simple interest formula

Write down the known variables and the simple interest formula

$$\begin{aligned} P &= 3450 \\ i &= 0{,}125 \\ n &= 5 \\ A &= P(1 + in) \end{aligned}$$

Substitute the values to determine the accumulated amount for the Wise Bank savings

account.

$$
\begin{aligned}
A &= 3450(1 + 0{,}125 \times 5) \\
&= \text{R } 5606{,}25
\end{aligned}
$$

Step 2: Calculation using the compound interest formula

Write down the known variables and the compound interest formula.

$$
\begin{aligned}
P &= 3450 \\
i &= 0{,}104 \\
n &= 5 \\
A &= P(1 + i)^n
\end{aligned}
$$

Substitute the values to determine the accumulated amount for the Grand Bank savings account.

$$
\begin{aligned}
A &= 3450(1 + 0{,}104)^5 \\
&= \text{R } 5658{,}02
\end{aligned}
$$

Step 3: Write the final answer

The Grand Bank savings account would give Sam the highest accumulated balance at the end of the 5 year period.

Worked example 2: Finding i

QUESTION

Bongani decides to put R 30 000 in an investment account. What compound interest rate must the investment account achieve for Bongani to double his money in 6 years? Give your answer correct to one decimal place.

SOLUTION

Step 1: Write down the known variables and the compound interest formula

$$
\begin{aligned}
A &= 60\ 000 \\
P &= 30\ 000 \\
n &= 6 \\
A &= P(1 + i)^n
\end{aligned}
$$

Step 2: Substitute the values and solve for i

$$\begin{aligned} 60\ 000 &= 30\ 000(1+i)^6 \\ \frac{60\ 000}{30\ 000} &= (1+i)^6 \\ 2 &= (1+i)^6 \\ \sqrt[6]{2} &= 1+i \\ \sqrt[6]{2} - 1 &= i \\ \therefore i &= 0{,}122\ldots \end{aligned}$$

Step 3: Write the final answer and comment

We round up to a rate of 12,3% p.a. to make sure that Bongani doubles his investment.

Exercise 9 – 1: Revision

1. Determine the value of an investment of R 10 000 at 12,1% p.a. simple interest for 3 years.
2. Calculate the value of R 8000 invested at 8,6% p.a. compound interest for 4 years.
3. Calculate how much interest John will earn if he invests R 2000 for 4 years at:
   a) 6,7% p.a. simple interest
   b) 5,4% p.a. compound interest
4. The value of an investment grows from R 2200 to R 3850 in 8 years. Determine the simple interest rate at which it was invested.
5. James had R 12 000 and invested it for 5 years. If the value of his investment is R 15 600, what compound interest rate did it earn?

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1. 237Y 2. 237Z 3. 2382 4. 2383 5. 2384

9.2 Simple and compound depreciation EMBJF

As soon as a new car leaves the dealership, its value decreases and it is considered "second-hand". Vehicles, equipment, machinery and other similar assets, all lose value over time as a result of usage and age. This loss in value is called depreciation. Assets that have a relatively long useful lifetime, such as machines, trucks, farming equipment etc., depreciate slower than assets like office equipment, computers, furniture etc. which need to be replaced more often and therefore depreciate more quickly.

Depreciation is used to calculate the value of a company's assets, which determines how much tax a company must pay. Companies can take depreciation into account as an expense, and thereby reduce their taxable income. A lower taxable income means that the company will pay less income tax to SARS (South African Revenue Service).

We can calculate two different kinds of depreciation: simple decay and compound decay. Decay is also a term used to describe a reduction or decline in value. Simple decay is also called straight-line depreciation and compound decay can also be referred to as reducing-balance depreciation. In the straight-line method the value of the asset is reduced by a constant amount each year, which is calculated on the principal amount. In reducing-balance depreciation we calculate the depreciation on the reduced value of the asset. This means that the value of an asset decreases by a different amount each year.

Investigation: Simple and compound depreciation

1. Mr. Sontange buys an Opel Fiesta for R 72 000. He expects that the value of the car will depreciate by R 6000 every year. He draws up a table to calculate the depreciated value of his Opel Fiesta.

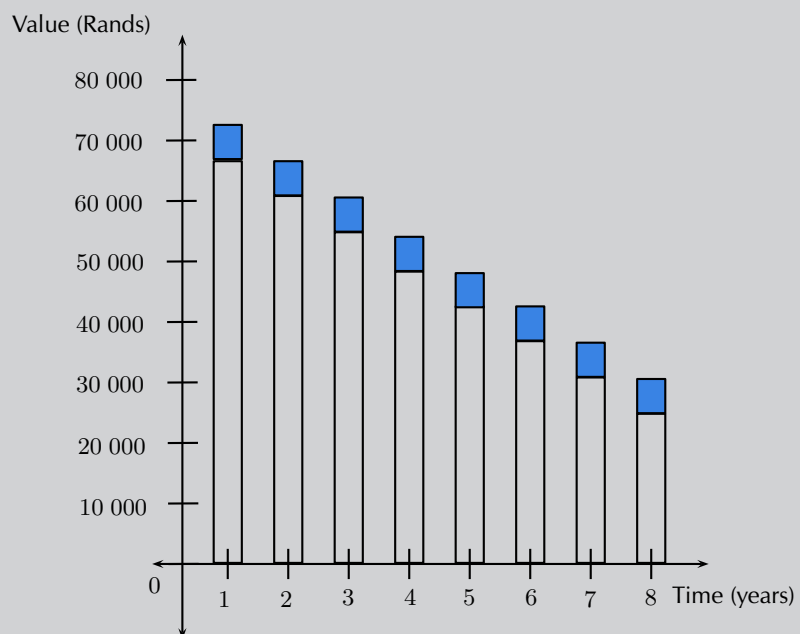
   Complete Mr. Sontange's table of values for the 7 year period:

Year	Value at beginning of year	Depreciation amount	Value at end of year
1	R 72 000	R 6000	R 66 000
2	R 66 000	R 6000	
3			
4			
5			
6			
7			

2. His son, David, does not agree that the value of the car will reduce by the same amount each year. David thinks that the car will depreciate by 10% every year.

   Complete David's table of values:

Year	Value at beginning of year	Depreciation amount	Value at end of year
1	R 72 000	R 7200	R 64 800
2	R 64 800	R 6480	
3			
4			
5			
6			
7			

3. Compare and discuss the results of the two different tables.
4. Consider the graph below, which represents Mr. Sontange's table of values:

a) Draw a similar graph using David's table of values.
b) Interpret the two graphs and discuss the differences between them.
c) Explain how the graphs can be used to determine the total depreciation in each case.
d) i. Draw two new graphs by plotting the maximum value of each bar.
   ii. Join the points with a line to show the general trend.
   iii. Is it mathematically correct to join these points? Explain your answer.

Worked example 3: Straight-line depreciation

QUESTION

A new smartphone costs R 6000 and depreciates at 22% p.a. on a straight-line basis. Determine the value of the smartphone at the end of each year over a 4 year period.

SOLUTION

Step 1: Calculate depreciation amount

$$\begin{aligned}\text{Depreciation} &= 6000 \times \frac{22}{100} \\ &= 1320\end{aligned}$$

Therefore the smartphone depreciates by R 1320 every year.

Step 2: Complete a table of values

Year	Value at beginning of year	Depreciation amount	Value at end of year
1	R 6000	R 1320	R 4680
2	R 4680	R 1320	R 3360
3	R 3360	R 1320	R 2040
4	R 2040	R 1320	R 720

We notice that

$$\text{Total depreciation} = P \times i \times n$$

where

$$\begin{aligned} P &= \text{principal amount} \\ i &= \text{interest rate written as a decimal} \\ n &= \text{time period in years}\end{aligned}$$

Therefore the depreciated value of the asset (also called the book value) can be calculated as:

$$A = P(1 - in)$$

Note the similarity to the simple interest formula $A = P(1 + in)$. Interest increases the value of the principal amount, whereas with simple decay, depreciation reduces the value of the principal amount.

Important: to get an accurate answer do all calculations in one step on your calculator. Do not round off answers in your calculations until the final answer. In the worked examples in this chapter, we use dots to show that the answer has not been rounded off. We always round the final answer to two decimal places (cents).

Worked example 4: Straight-line depreciation method

QUESTION

A car is valued at R 240 000. If it depreciates at 15% p.a. using straight-line depreciation, calculate the value of the car after 5 years.

SOLUTION

Step 1: Write down the known variables and the simple decay formula

$$\begin{aligned} P &= 240\ 000 \\ i &= 0{,}15 \\ n &= 5 \end{aligned}$$

$$A = P(1 - in)$$

Step 2: Substitute the values and solve for A

$$\begin{aligned} A &= 240\ 000(1 - 0{,}15 \times 5) \\ &= 240\ 000(0{,}25) \\ &= 60\ 000 \end{aligned}$$

Step 3: Write the final answer

At the end of 5 years, the car is worth R 60 000.

Worked example 5: Simple decay

QUESTION

A small business buys a photocopier for R 12 000. For the tax return the owner depreciates this asset over 3 years using a straight-line depreciation method. What amount will he fill in on his tax form at the end of each year?

SOLUTION

Step 1: Write down the known variables

The owner of the business wants the photocopier to have a book value of R 0 after 3 years.

$$\begin{aligned} A &= 0 \\ P &= 12\ 000 \\ n &= 3 \end{aligned}$$

Therefore we can calculate the annual depreciation as

$$\begin{aligned} \text{Depreciation} &= \frac{P}{n} \\ &= \frac{12\ 000}{3} \\ &= \text{R } 4000 \end{aligned}$$

Step 2: Determine the book value at the end of each year

$$\begin{aligned} \text{Book value end of first year} &= 12\ 000 - 4000 \\ &= \text{R } 8000 \end{aligned}$$

$$\begin{aligned} \text{Book value end of second year} &= 8000 - 4000 \\ &= \text{R } 4000 \end{aligned}$$

$$\begin{aligned} \text{Book value end of third year} &= 4000 - 4000 \\ &= \text{R } 0 \end{aligned}$$

Exercise 9 – 2: Simple decay

1. A business buys a truck for R 560 000. Over a period of 10 years the value of the truck depreciates to R 0 using the straight-line method. What is the value of the truck after 8 years?
2. Harry wants to buy his grandpa's donkey for R 800. His grandpa is quite pleased with the offer, seeing that it only depreciated at a rate of 3% per year using the straight-line method. Grandpa bought the donkey 5 years ago. What did grandpa pay for the donkey then?
3. Seven years ago, Rocco's drum kit cost him R 12 500. It has now been valued at R 2300. What rate of simple depreciation does this represent?
4. Fiona buys a DStv satellite dish for R 3000. Due to weathering, its value depreciates simply at 15% per annum. After how long will the satellite dish have a book value of zero?

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1. 2385 2. 2386 3. 2387 4. 2388

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Compound depreciation

EMBJH

Worked example 6: Reducing-balance depreciation

QUESTION

A second-hand farm tractor worth R 60 000 has a limited useful life of 5 years and depreciates at 20% p.a. on a reducing-balance basis. Determine the value of the tractor at the end of each year over the 5 year period.

SOLUTION

Step 1: Write down the known variables

$$P = 60\ 000$$
$$i = 0{,}2$$
$$n = 5$$

When we calculate depreciation using the reducing-balance method:

1. the depreciation amount changes for each year.
2. the depreciation amount gets smaller each year.
3. the book value at the end of a year becomes the principal amount for the next year.
4. the asset will always have some value (the book value will never equal zero).

Step 2: Complete a table of values

Year	Book value	Depreciation	Value at end of year
1	R 60 000	$60\ 000 \times 0{,}2 = 12\ 000$	R 48 000
2	R 48 000	$48\ 000 \times 0{,}2 = 9600$	R 38 400
3	R 38 400	$38\ 400 \times 0{,}2 = 7680$	R 30 720
4	R 30 720	$30\ 720 \times 0{,}2 = 6144$	R 24 576
5	R 24 576	$24\ 576 \times 0{,}2 = 4915{,}20$	R 19 660,80

Notice in the example above that we could also write the book value at the end of each year as:

$$\begin{aligned}
\text{Book value end of first year} &= 60\ 000(1 - 0{,}2) \\
\text{Book value end of second year} &= 48\ 000(1 - 0{,}2) = 60\ 000(1 - 0{,}2)^2 \\
\text{Book value end of third year} &= 38\ 400(1 - 0{,}2) = 60\ 000(1 - 0{,}2)^3 \\
\text{Book value end of fourth year} &= 30\ 720(1 - 0{,}2) = 60\ 000(1 - 0{,}2)^4 \\
\text{Book value end of fifth year} &= 24\ 576(1 - 0{,}2) = 60\ 000(1 - 0{,}2)^5
\end{aligned}$$

Using the formula for simple decay and the observed pattern in the calculation above, we obtain the following formula for compound decay:

$$A = P(1 - i)^n$$

where

$$\begin{aligned}
A &= \text{book value or depreciated value} \\
P &= \text{principal amount} \\
i &= \text{interest rate written as a decimal} \\
n &= \text{time period in years}
\end{aligned}$$

Again, notice the similarity to the compound interest formula $A = P(1 + i)^n$.

Worked example 7: Reducing-balance depreciation

QUESTION

The number of pelicans at the Berg river mouth is decreasing at a compound rate of 12% p.a. If there are currently 3200 pelicans in the wetlands of the Berg river mouth, what will the population be in 5 years?

SOLUTION

Step 1: Write down the known variables and the compound decay formula

$$\begin{aligned}
P &= 3200 \\
i &= 0{,}12 \\
n &= 5 \\
A &= P(1 - i)^n
\end{aligned}$$

Step 2: Substitute the values and solve for A

$$\begin{aligned}
A &= 3200(1 - 0{,}12)^5 \\
&= 3200(0{,}88)^5 \\
&= 1688{,}7421\ldots
\end{aligned}$$

Step 3: Write the final answer

In 5 years, the pelican population will be approximately 1689.

Worked example 8: Compound decay

QUESTION

1. A school buys a minibus for R 950 000, which depreciates at 13,5% per annum. Determine the value of the minibus after 3 years if the depreciation is calculated:

   a) on a straight-line basis.

   b) on a reducing-balance basis.

2. Which is the better option?

SOLUTION

Step 1: Write down known variables

$$
\begin{aligned}
P &= 950\ 000 \\
i &= 0{,}135 \\
n &= 3
\end{aligned}
$$

Step 2: Use the simple decay formula and solve for A

$$
\begin{aligned}
A &= 950\ 000(1 - 3 \times 0{,}135) \\
&= 950\ 000(0{,}865) \\
&= 565\ 250 \\
\therefore A &= \text{R } 565\ 250
\end{aligned}
$$

Step 3: Use the compound decay formula and solve for A

$$
\begin{aligned}
A &= 950\ 000(1 - 0{,}135)^3 \\
&= 950\ 000(0{,}865)^3 \\
&= 614\ 853{,}89 \\
\therefore A &= \text{R } 614\ 853{,}89
\end{aligned}
$$

Step 4: Interpret the answers

After a period of 3 years, the value of the minibus calculated on the straight-line method is less than the value of the minibus calculated on the reducing-balance method. The value of the minibus depreciated less on the reducing-balance basis because the amount of depreciation is calculated on a smaller amount every year, whereas the straight-line method is based on the full value of the minibus every year.

Worked example 9: Compound depreciation

QUESTION

Farmer Jack bought a tractor and it has depreciated by 20% p.a. on a reducing-balance basis. If the current value of the tractor is R 52 429, calculate how much Farmer Jack paid for his tractor if he bought it 7 years ago.

SOLUTION

Step 1: Write down known variables and compound decay formula

$$A = 52\ 429$$
$$i = 0{,}2$$
$$n = 7$$

$$A = P(1 - i)^n$$

Step 2: Substitute the values and solve for P

$$\begin{aligned} 52\ 429 &= P(1 - 0{,}2)^7 \\ &= P(0{,}8)^7 \\ \therefore P &= \frac{52\ 429}{(0{,}8)^7} \\ &= 250\ 000{,}95\ldots \end{aligned}$$

Step 3: Write the final answer

7 years ago, Farmer Jack paid R 250 000 for his tractor.

Exercise 9 – 3: Compound depreciation

1. Jwayelani buys a truck for R 89 000 and depreciates it by 9% p.a. using the compound depreciation method. What is the value of the truck after 14 years?
2. The number of cormorants at the Amanzimtoti river mouth is decreasing at a compound rate of 8% p.a. If there are now 10 000 cormorants, how many will there be in 18 years' time?
3. On January 1, 2008 the value of my Kia Sorento is R 320 000. Each year after that, the car's value will decrease 20% of the previous year's value. What is the value of the car on January 1, 2012?

4. The population of Bonduel decreases at a reducing-balance rate of 9,5% per annum as people migrate to the cities. Calculate the decrease in population over a period of 5 years if the initial population was 2 178 000.
5. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does it weigh after a month of 31 days?
6. Richard bought a car 15 years ago and it depreciated by 17% p.a. on a compound depreciation basis. How much did he pay for the car if it is now worth R 5256?

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1. 2389 2. 238B 3. 238C 4. 238D 5. 238F 6. 238G

Finding i EMBJJ

Worked example 10: Finding i for simple decay

QUESTION

After 4 years, the value of a computer is halved. Assuming simple decay, at what annual rate did it depreciate? Give your answer correct to two decimal places.

SOLUTION

Step 1: Write down known variables and simple decay formula

Let the value of the computer be x, therefore:

$$A = \frac{x}{2}$$
$$P = x$$
$$n = 4$$

$$A = P(1 - in)$$

Step 2: Substitute the values and solve for i

$$\frac{x}{2} = x(1 - 3i)$$
$$\frac{1}{2} = 1 - 3i$$
$$\therefore 3i = 1 - \frac{1}{2}$$
$$\therefore i = 0{,}1667$$

Step 3: Write the final answer

The computer depreciated at a rate of 16,67% p.a.

Worked example 11: Finding i for compound decay

QUESTION

Cristina bought a fridge at the beginning of 2009 for R 8999 and sold it at the end of 2011 for R 4500. At what rate did the value of her fridge depreciate assuming a reducing-balance method? Give your answer correct to two decimal places.

SOLUTION

Step 1: Write down known variables and compound decay formula

$$A = 4500$$
$$P = 8999$$
$$n = 3$$
$$A = P(1 - i)^n$$

Step 2: Substitute the values and solve for i

$$4500 = 8999(1 - i)^3$$
$$\frac{4500}{8999} = (1 - i)^3$$
$$\sqrt[3]{\frac{4500}{8999}} = 1 - i$$
$$\therefore i = 1 - \sqrt[3]{\frac{4500}{8999}}$$
$$= 0{,}206$$

Step 3: Write the final answer

Cristina's fridge depreciated at a rate of 20,6% p.a.

Exercise 9 – 4: Finding i

1. A machine costs R 45 000 and has a scrap value of R 9000 after 10 years. Determine the annual rate of depreciation if it is calculated on the reducing balance method.
2. After 15 years, an aeroplane is worth $\frac{1}{6}$ of its original value. At what annual rate was depreciation compounded?
3. Mr. Mabula buys furniture for R 20 000. After 6 years he sells the furniture for R 9300. Calculate the annual compound rate of depreciation of the furniture.
4. Ayanda bought a new car 7 years ago for double what it is worth today. At what yearly compound rate did her car depreciate?

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1. 238H 2. 238J 3. 238K 4. 238M

9.3 Timelines EMBJK

Interest can be compounded more than once a year. For example, an investment can be compounded monthly or quarterly. Below is a table of compounding terms and their corresponding numeric value (p). When amounts are compounded more than once per annum, we multiply the number of years by p and we also divide the interest rate by p.

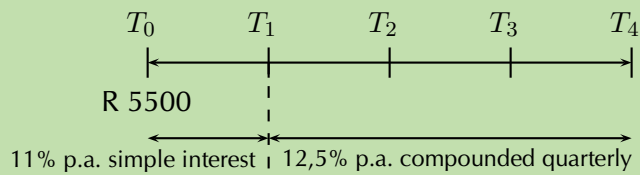
Term	p
yearly / annually	1
half-yearly / bi-annually	2
quarterly	4
monthly	12
weekly	52
daily	365

Worked example 12: Timelines

QUESTION

R 5500 is invested for a period of 4 years in a savings account. For the first year, the investment grows at a simple interest rate of 11% p.a. and then at a rate of 12,5% p.a. compounded quarterly for the rest of the period. Determine the value of the investment at the end of the 4 years.

SOLUTION

Step 1: Draw a timeline and write down known variables

In the timeline above, the intervals are given in years. For example, T_0 is the start of the investment, T_1 is the end of the first year and T_4 is the end of the fourth year.

Step 2: Use the simple interest formula to calculate A at T_1

$$\begin{aligned} A &= P(1 + in) \\ &= 5500(1 + 0{,}11) \\ &= \text{R } 6105 \end{aligned}$$

Step 3: Use the compound interest formula to calculate A at T_4

The investment is compounded quarterly, therefore:

$$\begin{aligned} n &= 3 \times 4 \\ &= 12 \\ \text{and } i &= \frac{0{,}125}{4} \end{aligned}$$

Also notice that the accumulated amount at the end of the first year becomes the principal amount at the beginning of the second year.

$$\begin{aligned} A &= P(1 + i)^n \\ &= 6105\left(1 + \frac{0{,}125}{4}\right)^{12} \\ &= \text{R } 8831{,}88 \end{aligned}$$

Step 4: Write the final answer

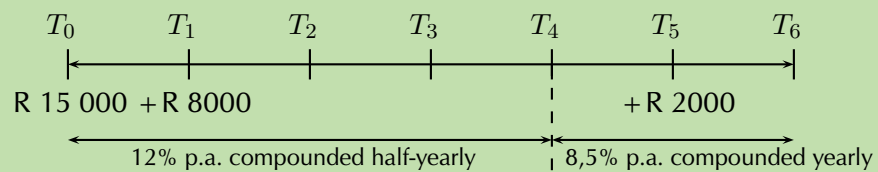
The value of the investment at the end of the 4 years is R 8831,88.

Worked example 13: Timelines

QUESTION

R 150 000 is deposited in an investment account for a period of 6 years at an interest rate of 12% p.a. compounded half-yearly for the first 4 years and then 8,5% p.a. compounded yearly for the rest of the period. A deposit of R 8000 is made into the account after the first year and then another deposit of R 2000 is made 5 years after the initial investment. Calculate the value of the investment at the end of the 6 years.

SOLUTION

Step 1: Draw a timeline and write down known variables

Remember to show when the additional deposits of R 8000 and R 2000 where made into the account. It is very important to note that the interest rate changes at T_4.

We break this question down into parts and consider each amount separately.

Step 2: The initial deposit at T_0

Between T_0 and T_4:

We notice that interest for the first 4 years is compounded half-yearly, therefore:

$$\begin{aligned} n_1 &= 4 \times 2 \\ &= 8 \\ \text{and } i_1 &= \frac{0{,}12}{2} \end{aligned}$$

Between T_4 and T_6:

$$\begin{aligned} n_2 &= 2 \\ \text{and } i_2 &= 0{,}085 \end{aligned}$$

Therefore the total growth of the initial deposit over the 6 years is:

$$\begin{aligned} A &= P(1+i_1)^{n_1}(1+i_2)^{n_2} \\ &= 150\,000\left(1+\frac{0{,}12}{2}\right)^8(1+0{,}085)^2 \end{aligned}$$

Step 3: The deposit at T_1

Between T_1 and T_4:

Interest on this deposit is compounded half-yearly for 3 years, therefore:

$$\begin{aligned} n_3 &= 3 \times 2 \\ &= 6 \\ \text{and } i_3 &= \frac{0{,}12}{2} \end{aligned}$$

Between T_4 and T_6:

$$\begin{aligned} n_4 &= 2 \\ \text{and } i_4 &= 0{,}085 \end{aligned}$$

Therefore the total growth of the deposit over the 5 years is:

$$\begin{aligned} A &= P(1 + i_3)^{n_3}(1 + i_4)^{n_4} \\ &= 8000\left(1 + \frac{0{,}12}{2}\right)^6 (1 + 0{,}085)^2 \end{aligned}$$

Step 4: The deposit at T_5

Accumulate interest for only 1 year:

$$\begin{aligned} A &= P(1 + i)^n \\ &= 2000(1 + 0{,}085)^1 \end{aligned}$$

Step 5: Determine the total calculation

To get as accurate an answer as possible, we do the the calculation on the calculator in one step. Using the memory and answer recall function on the calculator, we avoid rounding off until we get the final answer.

$$\begin{aligned} A &= 150\ 000\left(1 + \frac{0{,}12}{2}\right)^8 (1 + 0{,}085)^2 \\ &+ 8000\left(1 + \frac{0{,}12}{2}\right)^6 (1 + 0{,}085)^2 + 2000(1 + 0{,}085)^1 \\ &= \text{R } 296\ 977{,}00 \end{aligned}$$

Step 6: Write the final answer

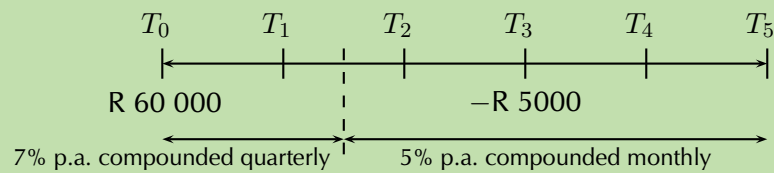
The value of the investment at the end of the 6 years is R 296 977,00.

Worked example 14: Timelines

QUESTION

R 60 000 is invested in an account which offers interest at 7% p.a. compounded quarterly for the first 18 months. Thereafter the interest rate changes to 5% p.a. compounded monthly. Three years after the initial investment, R 5000 is withdrawn from the account. How much will be in the account at the end of 5 years?

SOLUTION

Step 1: Draw a timeline and write down known variables

Remember to show when the withdrawal of R 5000 was taken out of the account. It is also important to note that the interest rate changes after 18 months ($T_{1\frac{1}{2}}$).

We break this question down into parts and consider each amount separately.

Step 2: The initial deposit at T_0

Interest for the first 1,5 years is compounded quarterly, therefore:

$$n_1 = 1{,}5 \times 4$$
$$= 6$$
$$\text{and } i_1 = \frac{0{,}07}{4}$$

Interest for the remaining 3,5 years is compounded monthly, therefore:

$$n_2 = 3{,}5 \times 12$$
$$= 42$$
$$\text{and } i_2 = \frac{0{,}05}{12}$$

Therefore the total growth of the initial deposit over the 5 years is:

$$A = P(1 + i_1)^{n_1}(1 + i_2)^{n_2}$$
$$= 60\ 000 \left(1 + \frac{0{,}07}{4}\right)^6 \left(1 + \frac{0{,}05}{12}\right)^{42}$$

Step 3: The withdrawal at T_3

We calculate the interest that the R 5000 would have earned if it had remained in the account:

$$\begin{aligned} n &= 2 \times 12 \\ &= 24 \\ \text{and } i &= \frac{0{,}05}{12} \end{aligned}$$

Therefore we have that:

$$\begin{aligned} A &= P(1+i)^n \\ &= 5000\left(1 + \frac{0{,}05}{12}\right)^{24} \end{aligned}$$

Step 4: Determine the total calculation

We subtract the withdrawal and the interest it would have earned from the accumulated amount at the end of the 5 years:

$$\begin{aligned} A &= 60\ 000\left(1+\frac{0{,}07}{4}\right)^{6}\left(1+\frac{0{,}05}{12}\right)^{42} - 5000\left(1+\frac{0{,}05}{12}\right)^{24} \\ &= \text{R } 73\ 762{,}19 \end{aligned}$$

Step 5: Write the final answer

The value of the investment at the end of the 5 years is R 73 762,19.

Exercise 9 – 5: Timelines

1. After a 20-year period Josh's lump sum investment matures to an amount of R 313 550. How much did he invest if his money earned interest at a rate of 13,65% p.a. compounded half yearly for the first 10 years, 8,4% p.a. compounded quarterly for the next five years and 7,2% p.a. compounded monthly for the remaining period?
2. Sindisiwe wants to buy a motorcycle. The cost of the motorcycle is R 55 000. In 1998 Sindisiwe opened an account at Sutherland Bank with R 16 000. Then in 2003 she added R 2000 more into the account. In 2007 Sindisiwe made another change: she took R 3500 from the account. If the account pays 6% p.a. compounded half-yearly, will Sindisiwe have enough money in the account at the end of 2012 to buy the motorcycle?
3. A loan has to be returned in two equal semi-annual instalments. If the rate of interest is 16% per annum, compounded semi-annually and each instalment is R 1458, find the sum borrowed.

4. A man named Phillip invests R 10 000 into an account at North Bank at an interest rate of 7,5% p.a. compounded monthly. After 5 years the bank changes the interest rate to 8% p.a. compounded quarterly. How much money will Phillip have in his account 9 years after the original deposit?

5. R 75 000 is invested in an account which offers interest at 11% p.a. compounded monthly for the first 24 months. Then the interest rate changes to 7,7% p.a. compounded half-yearly. If R 9000 is withdrawn from the account after one year and then a deposit of R 3000 is made three years after the initial investment, how much will be in the account at the end of 6 years?

6. Christopher wants to buy a computer, but right now he doesn't have enough money. A friend told Christopher that in 5 years the computer will cost R 9150. He decides to start saving money today at Durban United Bank. Christopher deposits R 5000 into a savings account with an interest rate of 7,95% p.a. compounded monthly. Then after 18 months the bank changes the interest rate to 6,95% p.a. compounded weekly. After another 6 months, the interest rate changes again to 7,92% p.a. compounded two times per year. How much money will Christopher have in the account after 5 years, and will he then have enough money to buy the computer?

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1. 238N 2. 238P 3. 238Q 4. 238R 5. 238S 6. 238T

9.4 Nominal and effective interest rates EMBJM

We have seen that although interest is quoted as a percentage per annum it can be compounded more than once a year. We therefore need a way of comparing interest rates. For example, is an annual interest rate of 8% compounded quarterly higher or lower than an interest rate of 8% p.a. compounded yearly?

Investigation: Nominal and effective interest rates

1. Calculate the accumulated amount at the end of one year if R 1000 is invested at 8% p.a. compound interest:

$$\begin{aligned} A &= P(1+i)^n \\ &= \ldots\ldots \end{aligned}$$

2. Calculate the value of R 1000 if it is invested for one year at 8% p.a. compounded:

Frequency	Calculation	Accumulated amount	Interest amount
half-yearly	$A = 1000\left(1 + \frac{0,08}{2}\right)^{1\times 2}$	R 1081,60	R 81,60
quarterly			
monthly			
weekly			
daily			

3. Use your results from the table above to calculate the effective rate that the investment of R 1000 earns in one year:

Frequency	Accumulated amount	Calculation	Effective interest rate
half-yearly	R 1081,60	$1081,60 = 1000(1+i)$ $\frac{1081,60}{1000} = 1+i$ $\frac{1081,60}{1000} - 1 = i$ $\therefore i = 0,0816$	$i = 8,16\%$
quarterly			
monthly			
weekly			
daily			

4. If you wanted to borrow R 10 000 from the bank, would it be better to pay it back at an interest rate of 22% p.a. compounded quarterly or 22% compounded monthly? Show your calculations.

An interest rate compounded more than once a year is called the nominal interest rate. In the investigation above, we determined that the nominal interest rate of 8% p.a. compounded half-yearly is actually an effective rate of 8,16% p.a.

Given a nominal interest rate $i^{(m)}$ compounded at a frequency of m times per year and the effective interest rate i, the accumulated amount calculated using both interest rates will be equal so we can write:

$$P(1+i) = P\left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$\therefore 1+i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

Worked example 15: Nominal and effective interest rates

QUESTION

Interest on a credit card is quoted as 23% p.a. compounded monthly. What is the effective annual interest rate? Give your answer correct to two decimal places.

SOLUTION

Step 1: Write down the known variables

Interest is being added monthly, therefore:

$$m = 12$$
$$i^{(12)} = 0{,}23$$

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

Step 2: Substitute values and solve for i

$$\begin{aligned} 1 + i &= \left(1 + \frac{0{,}23}{12}\right)^{12} \\ \therefore i &= 1 - \left(1 + \frac{0{,}23}{12}\right)^{12} \\ &= 25{,}59\% \end{aligned}$$

Step 3: Write the final answer

The effective interest rate is 25,59% per annum.

Worked example 16: Nominal and effective interest rates

QUESTION

Determine the nominal interest rate compounded quarterly if the effective interest rate is 9% per annum (correct to two decimal places).

SOLUTION

Step 1: Write down the known variables

Interest is being added quarterly, therefore:

$$m = 4$$
$$i = 0{,}09$$

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

Step 2: Substitute values and solve for $i^{(m)}$

$$\begin{aligned}
1 + 0{,}09 &= \left(1 + \frac{i^{(4)}}{4}\right)^4 \\
\sqrt[4]{1{,}09} &= 1 + \frac{i^{(4)}}{4} \\
\sqrt[4]{1{,}09} - 1 &= \frac{i^{(4)}}{4} \\
4\left(\sqrt[4]{1{,}09} - 1\right) &= i^{(4)} \\
\therefore i^{(4)} &= 8{,}71\%
\end{aligned}$$

Step 3: Write the final answer

The nominal interest rate is 8,71% p.a. compounded quarterly.

Exercise 9 – 6: Nominal and effect interest rates

1. Determine the effective annual interest rate if the nominal interest rate is:
    a) 12% p.a. compounded quarterly.
    b) 14,5% p.a. compounded weekly.
    c) 20% p.a. compounded daily.
2. Consider the following:
    - 16,8% p.a. compounded annually.
    - 16,4% p.a. compounded monthly.
    - 16,5% p.a. compounded quarterly.

    a) Determine the effective annual interest rate of each of the nominal rates listed above.
    b) Which is the best interest rate for an investment?
    c) Which is the best interest rate for a loan?

3. Calculate the effective annual interest rate equivalent to a nominal interest rate of 8,75% p.a. compounded monthly.

4. Cebela is quoted a nominal interest rate of 9,15% per annum compounded every four months on her investment of R 85 000. Calculate the effective rate per annum.

5. Determine which of the following would be the better agreement for paying back a student loan:

   a) 9,1% p.a. compounded quarterly.

   b) 9% p.a. compounded monthly.

   c) 9,3% p.a. compounded half-yearly.

6. Miranda invests R 8000 for 5 years for her son's study fund. Determine how much money she will have at the end of the period and the effective annual interest rate if the nominal interest of 6% is compounded:

	Calculation	Accumulated amount	Effective annual interest rate
yearly			
half-yearly			
quarterly			
monthly			

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1a. 238V 1b. 238W 1c. 238X 2. 238Y 3. 238Z 4. 2392
5. 2393 6. 2394

9.5 Summary EMBJN

See presentation: 2395 at www.everythingmaths.co.za

- Simple interest: $A = P(1 + in)$
- Compound interest: $A = P(1 + i)^n$
- Simple depreciation: $A = P(1 - in)$
- Compound depreciation: $A = P(1 - i)^n$
- Nominal and effective annual interest rates: $1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$

Exercise 9 – 7: End of chapter exercises

1. Thabang buys a Mercedes worth R 385 000 in 2007. What will the value of the Mercedes be at the end of 2013 if:

   a) the car depreciates at 6% p.a. straight-line depreciation.

   b) the car depreciates at 6% p.a. reducing-balance depreciation.

2. Greg enters into a 5-year hire-purchase agreement to buy a computer for R 8900. The interest rate is quoted as 11% per annum based on simple interest. Calculate the required monthly payment for this contract.

3. A computer is purchased for R 16 000. It depreciates at 15% per annum.

   a) Determine the book value of the computer after 3 years if depreciation is calculated according to the straight-line method.

   b) Find the rate according to the reducing-balance method that would yield, after 3 years, the same book value as calculated in the previous question.

4. Maggie invests R 12 500 for 5 years at 12% per annum compounded monthly for the first 2 years and 14% per annum compounded semi-annually for the next 3 years. How much will Maggie receive in total after 5 years?

5. Tintin invests R 120 000. He is quoted a nominal interest rate of 7,2% per annum compounded monthly.

   a) Calculate the effective rate per annum (correct to two decimal places).

   b) Use the effective rate to calculate the value of Tintin's investment if he invested the money for 3 years.

   c) Suppose Tintin invests his money for a total period of 4 years, but after 18 months makes a withdrawal of R 20 000, how much will he receive at the end of the 4 years?

6. Ntombi opens accounts at a number of clothing stores and spends freely. She gets herself into terrible debt and she cannot pay off her accounts. She owes Fashion World R 5000 and the shop agrees to let her pay the bill at a nominal interest rate of 24% compounded monthly.

   a) How much money will she owe Fashion World after two years?

   b) What is the effective rate of interest that Fashion World is charging her?

7. John invests R 30 000 in the bank for a period of 18 months. Calculate how much money he will have at the end of the period and the effective annual interest rate if the nominal interest of 8% is compounded:

	Calculation	Accumulated amount	Effective annual interest rate
yearly			
half-yearly			
quarterly			
monthly			
daily			

8. Convert an effective annual interest rate of 11,6% p.a. to a nominal interest rate compounded:

    a) half-yearly
    b) quarterly
    c) monthly

9. Joseph must sell his plot on the West Coast and he needs to get R 300 000 on the sale of the land. If the estate agent charges him 7% commission on the selling price, what must the buyer pay for the plot?

10. Mrs. Brown retired and received a lump sum of R 200 000. She deposited the money in a fixed deposit savings account for 6 years. At the end of the 6 years the value of the investment was R 265 000. If the interest on her investment was compounded monthly, determine:

    a) the nominal interest rate per annum
    b) the effective annual interest rate

11. R 145 000 is invested in an account which offers interest at 9% p.a. compounded half-yearly for the first 2 years. Then the interest rate changes to 4% p.a. compounded quarterly. Four years after the initial investment, R 20 000 is withdrawn. 6 years after the initial investment, a deposit of R 15 000 is made. Determine the balance of the account at the end of 8 years.

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1. 2396 2. 2397 3. 2398 4. 2399 5. 239B 6. 239C
7. 239D 8. 239F 9. 239G 10. 239H 11. 239J

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CHAPTER

Probability

10.1 Revision EMBJP

Terminology EMBJQ

Outcome: a single observation of an uncertain or random process (called an *experiment*). For example, when you accidentally drop a book, it might fall on its cover, on its back or on its side. Each of these options is a possible outcome.

Sample space of an experiment: the set of all possible outcomes of the experiment. For example, the sample space when you roll a single 6-sided die is the set $\{1; 2; 3; 4; 5; 6\}$. For a given experiment, there is exactly one sample space. The sample space is denoted by the letter S.

Event: a set of outcomes of an experiment. For example, during radioactive decay of 1 gramme of uranium-234, one possible event is that the number of alpha-particles emitted during 1 microsecond is between 225 and 235.

Probability of an event: a real number between 0 and 1 that describes how likely it is that the event will occur. A probability of 0 means the outcome of the experiment will never be in the event set. A probability of 1 means the outcome of the experiment will always be in the event set. When all possible outcomes of an experiment have equal chance of occurring, the probability of an event is the number of outcomes in the event set as a fraction of the number of outcomes in the sample space.

Relative frequency of an event: the number of times that the event occurs during experimental trials, divided by the total number of trials conducted. For example, if we flip a coin 10 times and it landed on heads 3 times, then the relative frequency of the heads event is $\frac{3}{10} = 0{,}3$.

Union of events: the set of all outcomes that occur in at least one of the events. For 2 events called A and B, we write the union as "A or B". Another way of writing the union is using set notation: $A \cup B$.

Intersection of events: the set of all outcomes that occur in all of the events. For 2 events called A and B, we write the intersection as "A and B". Another way of writing the intersection is using set notation: $A \cap B$.

Mutually exclusive events: events with no outcomes in common, that is $(A \text{ and } B) = \emptyset$. Mutually exclusive events can never occur simultaneously. For example the event that a number is even and the event that the same number is odd are mutually exclusive, since a number can never be both even and odd.

Complementary events: two mutually exclusive events that together contain all the outcomes in the sample space. For an event called A, we write the complement as "not A". Another way of writing the complement is as A'.

See video: 239K at www.everythingmaths.co.za

The **addition rule** (also called the sum rule) for any 2 events, A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

This rule relates the probabilities of 2 events with the probabilities of their union and intersection.

The **addition rule for 2 mutually exclusive events** is

$$P(A \text{ or } B) = P(A) + P(B)$$

This rule is a special case of the previous rule. Because the events are mutually exclusive, $P(A \text{ and } B) = 0$.

The **complementary rule** is

$$P(\text{not } A) = 1 - P(A)$$

This rule is a special case of the previous rule. Since A and (not A) are mutually exclusive, $P(A \text{ or } (\text{not } A)) = 1$.

See video: 239M at www.everythingmaths.co.za

Worked example 1: Events

QUESTION

You take all the hearts from a deck of cards. You then select a random card from the set of hearts. What is the sample space? What is the probability of each of the following events?

1. The card is the ace of hearts.
2. The card has a prime number on it.
3. The card has a letter of the alphabet on it.

SOLUTION

Step 1: Write down the sample space

Since we are considering only one suit from the deck of cards (the hearts), we need to write down only the letters and numbers on the cards. Therefore the sample space is

$$S = \{\text{A}; 2; 3; 4; 5; 6; 7; 8; 9; 10; \text{J}; \text{Q}; \text{K}\}$$

Step 2: Write down the event sets

- ace of hearts: $\{\text{A}\}$
- prime number: $\{2; 3; 5; 7\}$
- letter of alphabet: $\{\text{A}; \text{J}; \text{Q}; \text{K}\}$

Step 3: Compute the probabilities

The probability of an event is defined as the number of elements in the event set divided by the number of elements in the sample space. There are 13 elements in the sample space. So the probability of each event is

- ace of hearts: $\frac{1}{13}$
- prime number: $\frac{4}{13}$
- letter of alphabet: $\frac{4}{13}$

Worked example 2: Events

QUESTION

You roll two 6-sided dice. Let E be the event that the total number of dots on the dice is 10. Let F be the event that at least one die is a 3.

1. Write down the event sets for E and F.
2. Determine the probabilities for E and F.
3. Are E and F mutually exclusive? Why or why not?

SOLUTION

Step 1: Write down the sample space

The sample space of a single 6-sided die is just $\{1;2;3;4;5;6\}$. To get the sample space of two 6-sided dice, we have to take every possible pair of numbers from 1 to 6.

$$S = \left\{ \begin{array}{cccccc} (1;1) & (1;2) & (1;3) & (1;4) & (1;5) & (1;6) \\ (2;1) & (2;2) & (2;3) & (2;4) & (2;5) & (2;6) \\ (3;1) & (3;2) & (3;3) & (3;4) & (3;5) & (3;6) \\ (4;1) & (4;2) & (4;3) & (4;4) & (4;5) & (4;6) \\ (5;1) & (5;2) & (5;3) & (5;4) & (5;5) & (5;6) \\ (6;1) & (6;2) & (6;3) & (6;4) & (6;5) & (6;6) \end{array} \right\}$$

Step 2: Write down the events

For E the dice have to add to 10.

$$E = \{(4;6);(5;5);(6;4)\}$$

For F at least one die has to be 3.

$$F = \{(1;3);(3;1);(2;3);(3;2);(3;3);(4;3);(3;4);(5;3);(3;5);(6;3);(3;6)\}$$

Step 3: Compute the probabilities

The probability of an event is defined as the number of elements in the event set divided by the number of elements in the sample space. There are

- $6 \times 6 = 36$ outcomes in the sample space, S;
- 3 outcomes in event E; and
- 11 outcomes in event F.

Therefore

$$P(E) = \frac{3}{36} = \frac{1}{12}$$

and

$$P(F) = \frac{11}{36}$$

Step 4: Are they mutually exclusive

To test whether two events are mutually exclusive, we have to test whether their intersection is empty. Since E has no outcomes that contain a 3 on one of the dice, the intersection of E and F is empty: $(E \text{ and } F) = \emptyset$. This means that the events are mutually exclusive.

See video: 239N at www.everythingmaths.co.za

Exercise 10 – 1: Revision

1. A bag contains r red balls, b blue balls and y yellow balls. What is the probability that a ball drawn from the bag at random is yellow?
2. A packet has yellow and pink sweets. The probability of taking out a pink sweet is $\frac{7}{12}$. What is the probability of taking out a yellow sweet?
3. You flip a coin 4 times. What is the probability that you get 2 heads and 2 tails? Write down the sample space and the event set to determine the probability of this event.
4. In a class of 37 children, 15 children walk to school, 20 children have pets at home and 12 children who have a pet at home also walk to school. How many children walk to school and do not have a pet at home?
5. You roll two 6-sided dice and are interested in the following two events:
   - A: the sum of the dice equals 8
   - B: at least one of the dice shows a 1

   Show that these events are mutually exclusive.

6. You ask a friend to think of a number from 1 to 100. You then ask her the following questions:
   - Is the number even?
   - Is the number divisible by 7?

   How many possible numbers are less than 80 if she answered "yes" to both questions?

7. In a group of 42 pupils, all but 3 had a packet of chips or a Fanta or both. If 23 had a packet of chips and 7 of these also had a Fanta, what is the probability that one pupil chosen at random has:
   a) both chips and Fanta
   b) only Fanta

8. Tamara has 18 loose socks in a drawer. Eight of these are orange and two are pink. Calculate the probability that the first sock taken out at random is:
   a) orange
   b) not orange
   c) pink
   d) not pink
   e) orange or pink
   f) neither orange nor pink

9. A box contains coloured blocks. The number of blocks of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

   A block is selected randomly. What is the probability that the block will be:
   a) purple
   b) purple or white
   c) pink and orange
   d) not orange?

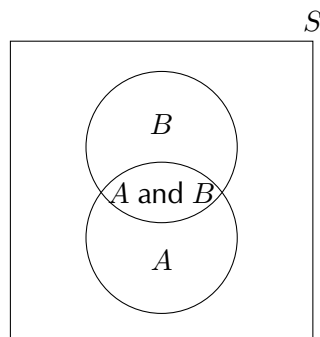
10. The surface of a soccer ball is made up of 32 faces. 12 faces are regular pentagons, each with a surface area of about 37 cm^2. The other 20 faces are regular hexagons, each with a surface area of about 56 cm^2.

    You roll the soccer ball. What is the probability that it stops with a pentagon touching the ground?

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 239P 2. 239Q 3. 239R 4. 239S 5. 239T 6. 239V
7. 239W 8. 239X 9. 239Y 10. 239Z

Venn diagrams EMBJS

A Venn diagram is used to show how events are related to one another. A Venn diagram can be very helpful when doing calculations with probabilities. In a Venn diagram each event is represented by a shape, often a circle or a rectangle. The region inside the shape represents the outcomes included in the event and the region outside the shape represents the outcomes that are not in the event.

A Venn diagram representing a sample space, S, as a square; and two events, A and B, as circles. The intersection of the two circles contains outcomes that are in both A and B.

Venn diagrams can be used in slightly different ways and it is important to notice the differences between them. The following 3 examples show how a Venn diagram is used to represent

- the outcomes included in each event;
- the number of outcomes in each event; and
- the probability of each event.

Worked example 3: Venn diagrams with outcomes

QUESTION

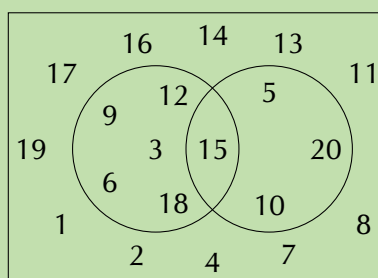
Choose a number between 1 and 20. Draw a Venn diagram to answer the following questions.

1. What is the probability that the number is a multiple of 3?
2. What is the probability that the number is a multiple of 5?
3. What is the probability that the number is a multiple of 3 or 5?
4. What is the probability that the number is a multiple of 3 and 5?

SOLUTION

Step 1: Draw a Venn diagram

The Venn diagram should show the sample space of all numbers from 1 to 20. It should also show an event set that contains all the multiples of 3, let $A = \{3; 6; 9; 12; 15; 18\}$,

and another event set that contains all the multiples of 5, let $B = \{5; 10; 15; 20\}$. Note that there is one shared outcome between these two events, namely 15.

Step 2: Compute probabilities

The probability of an event is the number of outcomes in the event set divided by the number of outcomes in the sample space. There are 20 outcomes in the sample space.

1. Since there are 6 outcomes in the multiples of 3 event set, the probability of a multiple of 3 is $P(A) = \frac{6}{20} = \frac{3}{10}$.
2. Since there are 4 outcomes in the multiples of 5 event set, the probability of a multiple of 5 is $P(B) = \frac{4}{20} = \frac{1}{5}$.
3. The event that the number is a multiple of 3 or 5 is the union of the above two event sets. There are 9 elements in the union of the event sets, so the probability is $\frac{9}{20}$.
4. The event that the number is a multiple of 3 and 5 is the intersection of the two event sets. There is 1 element in the intersection of the event sets, so the probability is $\frac{1}{20}$.

Worked example 4: Venn diagrams with counts

QUESTION

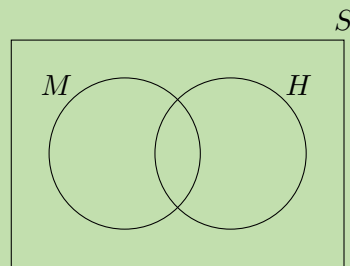
In a group of 50 learners, 35 take Mathematics and 30 take History, while 12 take neither of the two subjects. Draw a Venn diagram representing this information. If a learner is chosen at random from this group, what is the probability that he takes both Mathematics and History?

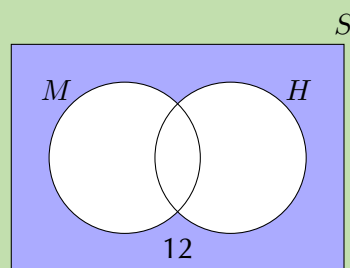
SOLUTION

Step 1: Draw outline of Venn diagram

There are 2 events in this question, namely

- M: that a learner takes Mathematics; and
- H: that a learner takes History.

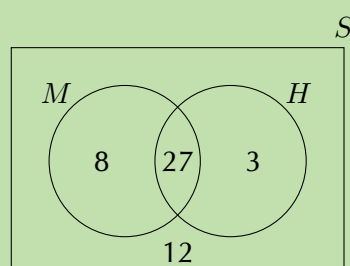
We need to do some calculations before drawing the full Venn diagram, but with the information above we can already draw the outline.

Step 2: Write down sizes of the event sets, their union and intersection

We are told that 12 learners take neither of the two subjects. Graphically we can represent this as:

Since there are 50 elements in the sample space, we can see from this figure that there are $50 - 12 = 38$ elements in $(M \text{ or } H)$. So far we know

- $n(M) = 35$
- $n(H) = 30$
- $n(M \text{ or } H) = 38$

From the addition rule,

$$\begin{aligned} n(M \text{ or } H) &= n(M) + n(H) - n(M \text{ and } H) \\ \therefore n(M \text{ and } H) &= 35 + 30 - 38 \\ &= 27 \end{aligned}$$

Step 3: Draw the final Venn diagram

Worked example 5: Venn diagrams with probabilities

QUESTION

Draw a Venn diagram to represent the same information as in the previous example, except showing the probabilities of the different events, rather than the counts.

If a learner is chosen at random from this group, what is the probability that she takes both Mathematics and History?

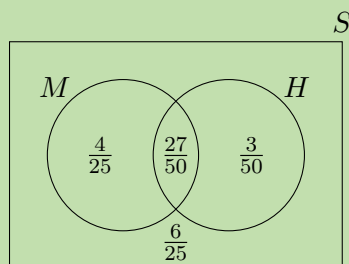
SOLUTION

Step 1: Use counts to compute probabilities

Since there are 50 elements (learners) in the sample space, we can compute the probability of any event by dividing the size of the event set by 50. This gives the following probabilities:

- $P(M) = \frac{35}{50} = \frac{7}{10}$
- $P(H) = \frac{30}{50} = \frac{3}{5}$
- $P(M \text{ or } H) = \frac{38}{50} = \frac{19}{25}$
- $P(M \text{ and } H) = \frac{27}{50}$

Step 2: Draw the Venn diagram

Next we replace each count from the Venn diagram in the previous example with a probability.

Step 3: Find the answer

The probability that a random learner will take both Mathematics and History is $P(M \text{ and } H) = \frac{27}{50}$.

See video: 23B2 at www.everythingmaths.co.za

Exercise 10 – 2: Venn diagram revision

1. Given the following information:
   - $P(A) = 0{,}3$
   - $P(B \text{ and } A) = 0{,}2$
   - $P(B) = 0{,}7$

   First draw a Venn diagram to represent this information. Then compute the value of $P(B \text{ and } (\text{not } A))$.

2. You are given the following information:
   - $P(A) = 0{,}5$
   - $P(A \text{ and } B) = 0{,}2$
   - $P(\text{not } B) = 0{,}6$

   Draw a Venn diagram to represent this information and determine $P(A \text{ or } B)$.

3. A study was undertaken to see how many people in Port Elizabeth owned either a Volkswagen or a Toyota. 3% owned both, 25% owned a Toyota and 60% owned a Volkswagen. What percentage of people owned neither car?

4. Let S denote the set of whole numbers from 1 to 15, X denote the set of even numbers from 1 to 15 and Y denote the set of prime numbers from 1 to 15. Draw a Venn diagram depicting S, X and Y.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 23B3 2. 23B4 3. 23B5 4. 23B6

 www.everythingmaths.co.za m.everythingmaths.co.za

10.2 Dependent and independent events EMBJT

Sometimes the presence or absence of one event tells us something about other events. We call events dependent if knowing whether one of them happened tells us something about whether the others happened. Independent events give us no information about one another; the probability of one event occurring does not affect the probability of the other events occurring.

DEFINITION: *Independent events*

Two events, A and B are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

At first it might not be clear why we should call events that satisfy the equation above independent. We will explore this further using a number of examples.

Investigation: Independence

Roll a single 6-sided die and consider the following two events:

- E: you get an even number
- T: you get a number that is divisible by three

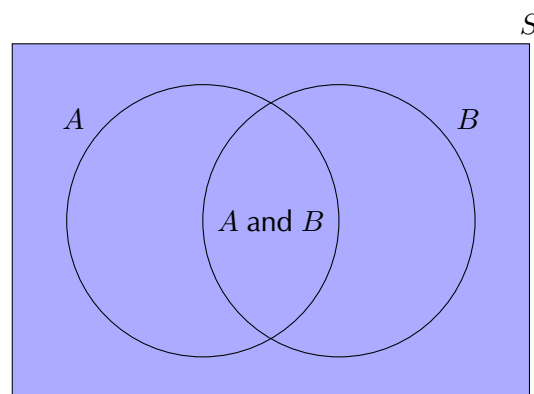
Now answer the following questions:

- What is the probability of E?
- What is the probability of getting an even number if you are told that the number was also divisible by three?
- Does knowing that the number was divisible by three change the probability that the number was even?

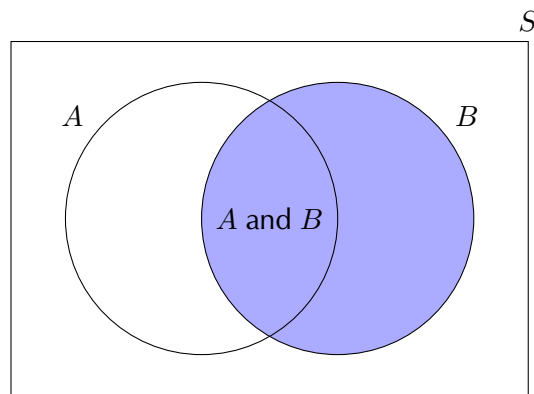
Are the events E and T dependent or independent according to the definition (hint: compute the probabilities in the definition of independence)?

See video: 23B7 at www.everythingmaths.co.za

So, why do we call it **independence** when $P(A \text{ and } B) = P(A) \times P(B)$? For two events, A and B, independence means that knowing the outcome of B does not affect the probability of A.

Consider the following Venn diagram.

The probability of A is the ratio between the number of outcomes in A and the number of outcomes in the sample space, S.

$$P(A) = \frac{n(A)}{n(S)}$$

Now, let's say that we **know** that event B happened. How does this affect the probability of A? Here is how the Venn diagram changes:

A lot of the possible outcomes (all of the outcomes outside B) are now out of the picture, because we know that they did not happen. Now the probability of A happening, given that we know that B happened, is the ratio between the size of the region where A is present (A and B) and the size of all possible events (B).

$$P(A \text{ if we know } B) = \frac{n(A \text{ and } B)}{n(B)}$$

If $P(A) = P(A \text{ if we know } B)$ we call them independent, because knowing B does not change the probability of A.

With some algebra, we can prove that this statement of independence is the same as the definition of independence that we saw at the beginning of this section. For independent events

$$P(A \text{ and } B) = P(A) \times P(B)$$

This is equivalent to

$$\begin{aligned} P(A) &= P(A \text{ and } B) \div P(B) \\ &= \frac{n(A \text{ and } B)}{n(S)} \div \frac{n(B)}{n(S)} \\ &= \frac{n(A \text{ and } B)}{n(B)} \\ &= P(A \text{ if we know } B) \end{aligned}$$

That is why we call events independent!

(For enrichment only):

The ratio

$$\frac{P(A \text{ and } B)}{P(B)}$$

is called a **conditional probability** and written using the notation $P(A \mid B)$. This notation is read as "the probability of A given B."

If (and only if) A and B are independent: $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$. Try to prove this using the definition of independence.

Worked example 6: Independent and dependent events

QUESTION

A bag contains 5 red and 5 blue balls. We remove a random ball from the bag, record its colour and put it back into the bag. We then remove another random ball from the bag and record its colour.

1. What is the probability that the first ball is red?
2. What is the probability that the second ball is blue?
3. What is the probability that the first ball is red and the second ball is blue?
4. Are the first ball being red and the second ball being blue independent events?

SOLUTION

Step 1: Probability of a red ball first

Since there are a total of 10 balls, of which 5 are red, the probability of getting a red ball is

$$P(\text{first ball red}) = \frac{5}{10} = \frac{1}{2}$$

Step 2: Probability of a blue ball second

The problem states that the first ball is placed back into the bag before we take the second ball. This means that when we draw the second ball, there are again a total of 10 balls in the bag, of which 5 are blue. Therefore the probability of drawing a blue ball is

$$P(\text{second ball blue}) = \frac{5}{10} = \frac{1}{2}$$

Step 3: Probability of red first and blue second

When drawing two balls from the bag, there are 4 possibilities. We can get

- a red ball and then another red ball;
- a red ball and then a blue ball;
- a blue ball and then a red ball;
- a blue ball and then another blue ball.

We want to know the probability of the second outcome, where we have to get a red ball first. Since there are 5 red balls and 10 balls in total, there are $\frac{5}{10}$ ways to get a red ball first. Now we put the first ball back, so there are again 5 red balls and 5 blue balls in the bag. Therefore there are $\frac{5}{10}$ ways to get a blue ball second if the first ball was red. This means that there are

$$\frac{5}{10} \times \frac{5}{10} = \frac{25}{100}$$

ways to get a red ball first and a blue ball second. So, the probability of getting a red ball first and a blue ball second is $\frac{1}{4}$.

Step 4: Dependent or independent?

According to the definition, events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

In this problem:

- $P(\text{first ball red}) = \frac{1}{2}$
- $P(\text{second ball blue}) = \frac{1}{2}$
- $P(\text{first ball red and second ball blue}) = \frac{1}{4}$

Since $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$, the events are independent.

See video: 23B8 at www.everythingmaths.co.za

Worked example 7: Independent and dependent events

QUESTION

In the previous example, we picked a random ball and put it back into the bag before continuing. This is called **sampling with replacement**. In this example, we will follow the same process, except that we will not put the first ball back into the bag. This is called **sampling without replacement**.

So, from a bag with 5 red and 5 blue balls, we remove a random ball and record its colour. Then, without putting back the first ball, we remove another random ball from the bag and record its colour.

1. What is the probability that the first ball is red?
2. What is the probability that the second ball is blue?
3. What is the probability that the first ball is red and the second ball is blue?
4. Are the first ball being red and the second ball being blue independent events?

SOLUTION

Step 1: Count the number of outcomes

We will look directly at the number of possible ways in which we can get the 4 possible outcomes when removing 2 balls. In the previous example, we saw that the 4 possible outcomes are

- a red ball and then another red ball;
- a red ball and then a blue ball;
- a blue ball and then a red ball;
- a blue ball and then another blue ball.

For the first outcome, we have to get a red ball first. Since there are 5 red balls and 10 balls in total, there are $\frac{5}{10}$ ways to get a red ball first. After we have taken out a red ball, there are now 4 red balls and 5 blue balls left. Therefore there are $\frac{4}{9}$ ways to get a red ball second if the first ball was also red. This means that there are

$$\frac{5}{10} \times \frac{4}{9} = \frac{20}{90}$$

ways to get a red ball first and a red ball second. The probability of the first outcome is $\frac{2}{9}$.

For the second outcome, we have to get a red ball first. As in the first outcome, there are $\frac{5}{10}$ ways to get a red ball first; and there are now 4 red balls and 5 blue balls left. Therefore there are $\frac{5}{9}$ ways to get a blue ball second if the first ball was red. This means that there are

$$\frac{5}{10} \times \frac{5}{9} = \frac{25}{90}$$

ways to get a red ball first and a blue ball second. The probability of the second outcome is $\frac{5}{18}$.

We can compute the probabilities of the third and fourth outcomes in the same way as the first two, but there is an easier way. Notice that there are only 2 types of ball and that there are exactly equal numbers of them at the start. This means that the problem is completely symmetric in red and blue. We can use this symmetry to compute the probabilities of the other two outcomes.

In the third outcome, the first ball is blue and the second ball is red. Because of symmetry this outcome must have the same probability as the second outcome (when the first ball is red and the second ball is blue). Therefore the probability of the third outcome is $\frac{5}{18}$.

In the fourth outcome, the first and second balls are both blue. From symmetry, this outcome must have the same probability as the first outcome (when both balls are red). Therefore the probability of the fourth outcome is $\frac{2}{9}$.

To summarise, these are the possible outcomes and their probabilities:

- first ball red and second ball red: $\frac{2}{9}$;
- first ball red and second ball blue: $\frac{5}{18}$;
- first ball blue and second ball red: $\frac{5}{18}$;
- first ball blue and second ball blue: $\frac{2}{9}$.

Step 2: Probability of a red ball first

To determine the probability of getting a red ball on the first draw, we look at all of the outcomes that contain a red ball first. These are

- a red ball and then another red ball;
- a red ball and then a blue ball.

The probability of the first outcome is $\frac{2}{9}$ and the probability of the second outcome is $\frac{5}{18}$. By adding these two probabilities, we see that the probability of getting a red ball first is

$$P(\text{first ball red}) = \frac{2}{9} + \frac{5}{18} = \frac{1}{2}$$

This is the same as in the previous exercise, which should not be too surprising since the probability of the first ball being red is not affected by whether or not we put it back into the bag before drawing the second ball.

Step 3: Probability of a blue ball second

To determine the probability of getting a blue ball on the second draw, we look at all of the outcomes that contain a blue ball second. These are

- a red ball and then a blue ball;
- a blue ball and then another blue ball.

The probability of the first outcome is $\frac{5}{18}$ and the probability of the second outcome is $\frac{2}{9}$. By adding these two probabilities, we see that the probability of getting a blue ball second is

$$P(\text{second ball blue}) = \frac{5}{18} + \frac{2}{9} = \frac{1}{2}$$

This is also the same as in the previous exercise! You might find it surprising that the probability of the second ball is not affected by whether or not we replace the first ball. The reason why this probability is still $\frac{1}{2}$ is that we are computing the probability that the second ball is blue without knowing the colour of the first ball. Because there are only two equal possibilities for the second ball (red and blue) and because we don't know whether the first ball is red or blue, there is an equal chance that the second ball will be one colour or the other.

Step 4: Probability of red first and blue second

We have already calculated the probability that the first ball is red and the second ball is blue. It is $\frac{5}{18}$.

Step 5: Dependent or independent?

According to the definition, events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

In this problem:

- $P(\text{first ball red}) = \frac{1}{2}$
- $P(\text{second ball blue}) = \frac{1}{2}$
- $P(\text{first ball red and second ball blue}) = \frac{5}{18}$

Since $\frac{5}{18} \neq \frac{1}{2} \times \frac{1}{2}$, the events are dependent.

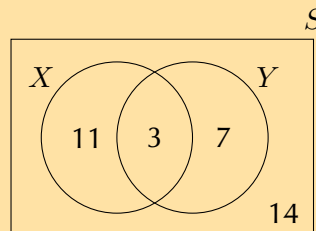
WARNING!

Just because two events are mutually exclusive does not necessarily mean that they are independent. To test whether events are mutually exclusive, always check that $P(A \text{ and } B) = 0$. To test whether events are independent, always check that $P(A \text{ and } B) = P(A) \times P(B)$. See the exercises below for examples of events that are mutually exclusive and independent in different combinations.

Exercise 10 – 3: Dependent and independent events

1. Use the following Venn diagram to determine whether events X and Y are

   a) mutually exclusive or not mutually exclusive;

   b) dependent or independent.

2. Of the 30 learners in a class 17 have black hair, 11 have brown hair and 2 have red hair. A learner is selected from the class at random.

   a) What is the probability that the learner has black hair?

   b) What is the probability that the learner has brown hair?

   c) Are these two events mutually exclusive?

   d) Are these two events independent?

3. $P(M) = 0{,}45$; $P(N) = 0{,}3$ and $P(M \text{ or } N) = 0{,}615$. Are the events M and N mutually exclusive, independent or neither mutually exclusive nor independent?

4. (For enrichment)

   Prove that if event A and event B are mutually exclusive with $P(A) \neq 0$ and $P(B) \neq 0$, then A and B are always dependent.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 23B9 2. 23BB 3. 23BC 4. 23BD

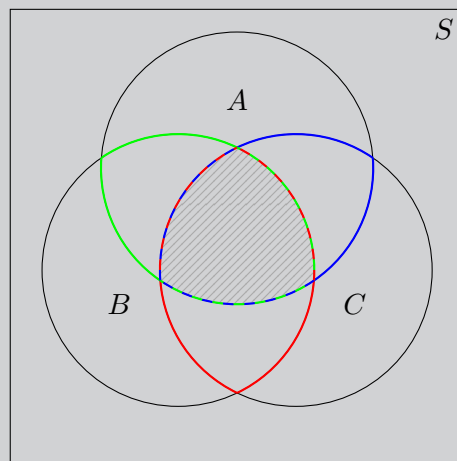
10.3 More Venn diagrams

EMBJV

In the rest of this chapter we will look at tools and techniques for working with probability problems.

When working with more complex problems, we can have three or more events that intersect in various ways. To solve these problems, we usually want to count the number (or percentage) of outcomes in an event, or a combination of events. Venn diagrams are a useful tool for recording and visualising the counts.

Investigation: Venn diagram for 3 events

The diagram below shows a general Venn diagram for 3 events.

Write down the sets corresponding to each of the three coloured regions and also to the shaded region. Remember that the intersections between circles represent the intersections between the different events.

What is the event for

- the red region;
- the green region;
- the blue region; and
- the shaded region?

Worked example 8: Venn diagram for 3 events

QUESTION

Draw a Venn diagram that shows the following sample space and events:

- S: all the integers from 1 to 30
- P: prime numbers
- M: multiples of 3
- F: factors of 30

SOLUTION

Step 1: Write down the sample space and event sets

The sample space contains all the positive integers up to 30.

$$S = \{1; 2; 3; \ldots; 30\}$$

The prime numbers between 1 and 30 are

$$P = \{2; 3; 5; 7; 11; 13; 17; 19; 23; 29\}$$

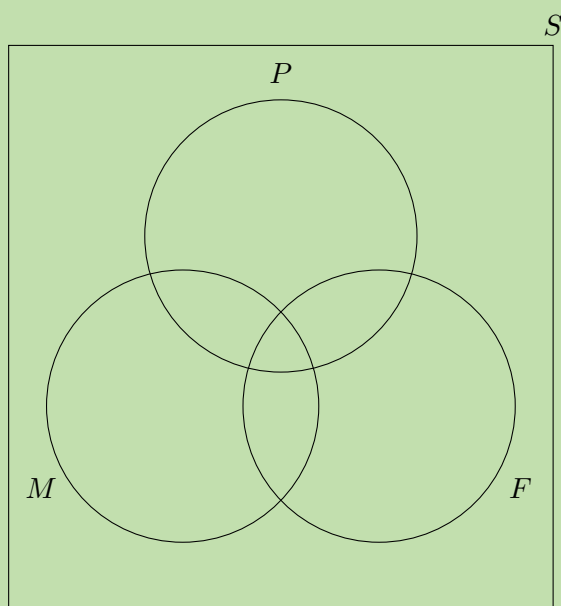
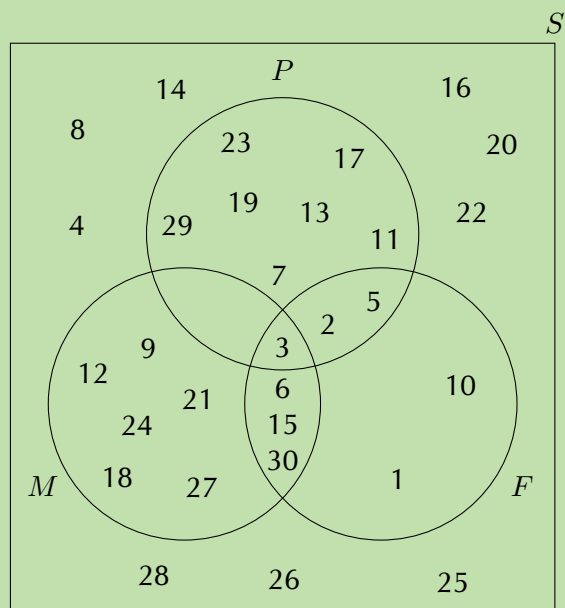
The multiples of 3 between 1 and 30 are

$$M = \{3; 6; 9; 12; 15; 18; 21; 24; 27; 30\}$$

The factors of 30 are

$$F = \{1; 2; 3; 5; 6; 10; 15; 30\}$$

Step 2: Draw the outline of the Venn diagram

There are 3 events, namely P, M and F, and the sample space, S. Put this information on a Venn diagram:

Step 3: Place the outcomes in the appropriate event sets

Worked example 9: Venn diagram for 3 events

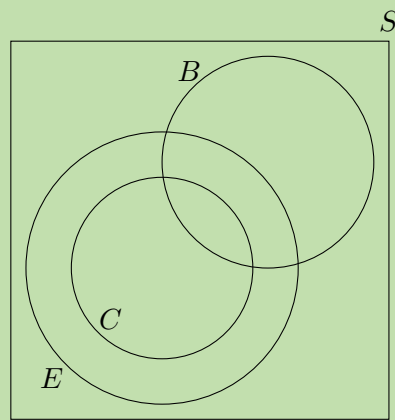
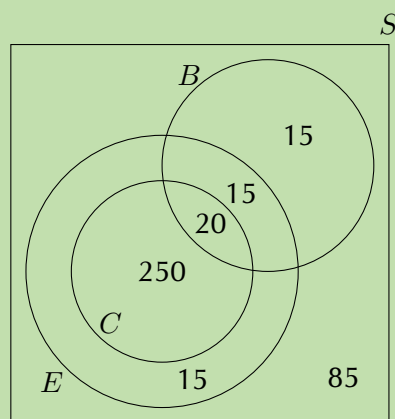
QUESTION

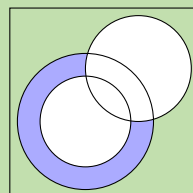
At Dawnview High there are 400 Grade 11 learners. 270 do Computer Science, 300 do English and 50 do Business studies. All those doing Computer Science do English, 20 take Computer Science and Business studies and 35 take English and Business studies. Using a Venn diagram, calculate the probability that a pupil drawn at random will take:

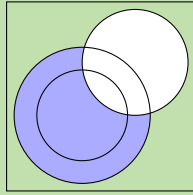
1. English, but not Business studies or Computer Science
2. English but not Business studies
3. English or Business studies but not Computer Science
4. English or Business studies

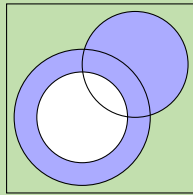
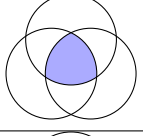
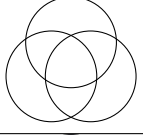
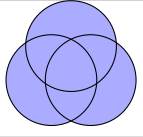
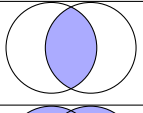
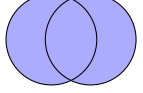
SOLUTION

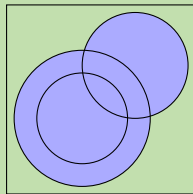
Step 1: Draw the outline of the Venn diagram

We need to be careful with this problem. In the question statement we are told that all the learners who do Computer Science also do English. This means that the circle for Computer Science on the Venn diagram needs to be inside the circle for English.

Step 2: Fill in the counts on the Venn diagram

Step 3: Compute probabilities

To find the number of learners taking English, but not Business studies or Computer Science, we need to look at this region of the Venn diagram:

The count in this region is 15 and there are a total of 400 learners in the grade. Therefore the probability that a learner will take English but not Business studies or Computer Science is $\frac{15}{400} = \frac{3}{80}$.

To find the number of learners taking English but not Business studies, we need to look at this region of the Venn diagram:

The count in this region is 265. Therefore the probability that a learner will take English but not Business studies is $\frac{265}{400} = \frac{53}{80}$.

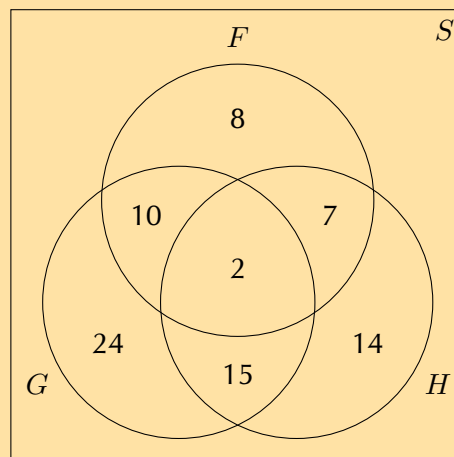
To find the number of learners taking English or Business studies but not Computer Science, we need to look at this region of the Venn diagram:

The count in this region is 45. Therefore the probability that a learner will take English or Business studies but not Computer Science is $\frac{45}{400} = \frac{9}{80}$.

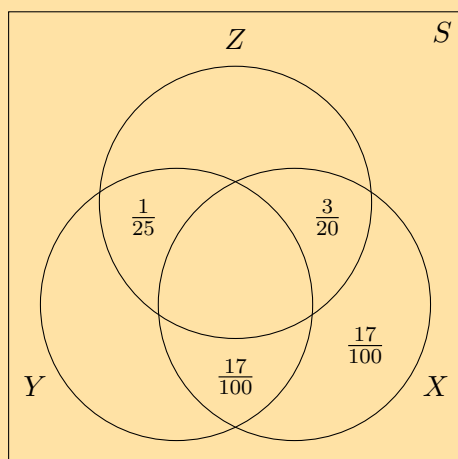
To find the number of learners taking English or Business studies, we need to look at this region of the Venn diagram:

The count in this region is 315. Therefore the probability that a learner will take English or Business studies is $\frac{315}{400} = \frac{63}{80}$.

There are some words that tell you which part of the Venn diagram should be filled in. The following table summarises the most important ones:

Words	Symbols	Venn diagram
"all"	A and B and C / $A \cap B \cap C$	
"none"		
"at least one"	A or B or C / $A \cup B \cup C$	
"both A and B"	A and B / $A \cap B$	
"A or B"	A or B / $A \cup B$	

Exercise 10 – 4: Venn diagrams

1. Use the Venn diagram below to answer the following questions. Also given: $n(S) = 120$.

   a) Compute $P(F)$.
   b) Compute $P(G \text{ or } H)$.
   c) Compute $P(F \text{ and } G)$.
   d) Are F and G dependent or independent?

2. The Venn diagram below shows the probabilities of 3 events. Complete the Venn diagram using the additional information provided.

- $P(Z \text{ and } (\text{not } Y)) = \frac{31}{100}$
- $P(Y \text{ and } X) = \frac{23}{100}$
- $P(Y) = \frac{39}{100}$

After completing the Venn diagram, compute the following:

$$P\left(Z \text{ and not } (X \text{ or } Y)\right)$$

3. There are 79 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41; those who take History is 36; and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

   a) Draw a Venn diagram to illustrate all this information.
   b) How many learners take Maths and Geography but not History?
   c) How many learners take Geography only?
   d) How many learners take all three subjects?

4. Draw a Venn diagram with 3 mutually exclusive events. Use the diagram to show that for 3 mutually exclusive events, A, B and C, the following is true:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$$

This is the addition rule for 3 mutually exclusive events.

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1. 23BF 2. 23BG 3. 23BH 4. 23BJ

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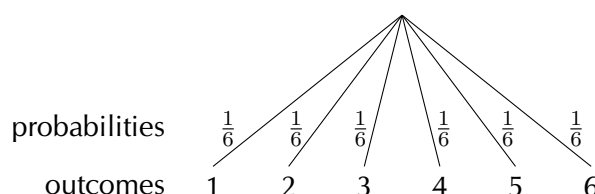
Tree diagrams are useful for organising and visualising the different possible outcomes of a sequence of events. For each possible outcome of the first event, we draw a line where we write down the probability of that outcome and the state of the world if that outcome happened. Then, for each possible outcome of the second event we do the same thing.

Below is an example of a simple tree diagram, showing the possible outcomes of rolling a 6-sided die.

Note that each outcome (the numbers 1 to 6) is shown at the end of a line; and that the probability of each outcome (all $\frac{1}{6}$ in this case) is shown shown on a line. The probabilities have to add up to 1 in order to cover all of the possible outcomes. In the examples below, we will see how to draw tree diagrams with multiple events and how to compute probabilities using the diagrams.

Earlier in this chapter you learned about dependent and independent events. Tree diagrams are very helpful for analysing dependent events. A tree diagram allows you to show how each possible outcome of one event affects the probabilities of the other events.

Tree diagrams are not so useful for independent events since we can just multiply the probabilities of separate events to get the probability of the combined event. Remember that for independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

So if you already know that events are independent, it is usually easier to solve a problem without using tree diagrams. But if you are uncertain about whether events are independent or if you know that they are not, you should use a tree diagram.

Worked example 10: Drawing a tree diagram

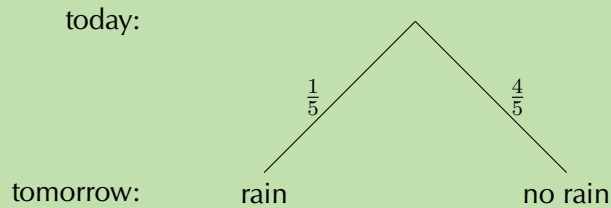
QUESTION

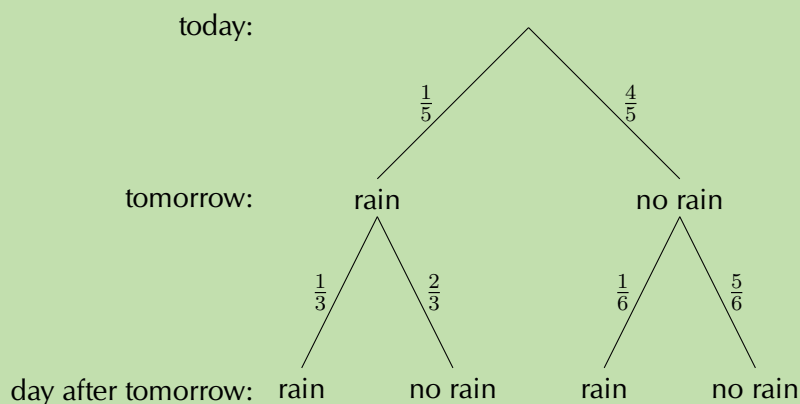
If it rains on a given day, the probability that it rains the next day is $\frac{1}{3}$. If it does not rain on a given day, the probability that it rains the next day is $\frac{1}{6}$. The probability that it will rain tomorrow is $\frac{1}{5}$. What is the probability that it will rain the day after tomorrow? Draw a tree diagram of all the possibilities to determine the answer.

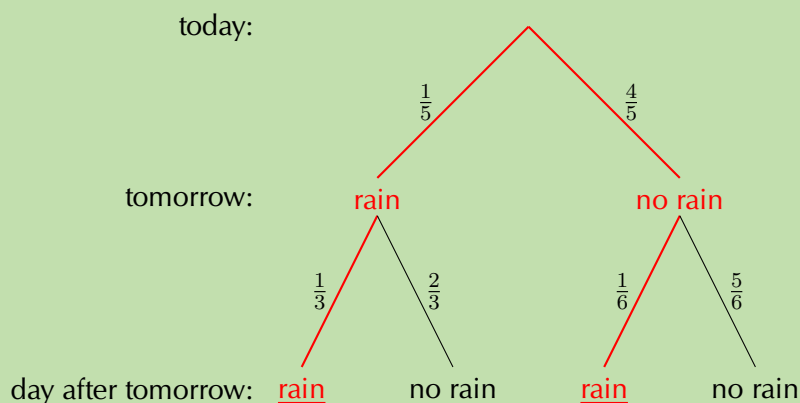
SOLUTION

Step 1: Draw the first level of the tree diagram

Before we can determine what happens on the day after tomorrow, we first have to determine what might happen tomorrow. We are told that there is a $\frac{1}{5}$ probability that

it will rain tomorrow. Here is how to represent this information using a tree diagram:

Step 2: Draw the second level of the tree diagram

We are also told that if it **does** rain on one day, there is a $\frac{1}{3}$ probability that it will also rain on the following day. On the other hand, if it **does not** rain on one day, there is only a $\frac{1}{6}$ probability that it will also rain on the following day. Using this information we complete the tree diagram:

Step 3: Compute the probability

We are asked what the probability is that it will rain the day after tomorrow. On the tree diagram above we can see that there are 2 situations where it rains on the day after tomorrow. They are marked in red below.

To get the probability for the first situation (that it rains tomorrow and the day after tomorrow) we have to multiply the probabilies along the first red line.

$$\begin{aligned}&P(\text{rain tomorrow and rain day after tomorrow})\\ =&\frac{1}{5}\times\frac{1}{3}\\ =&\frac{1}{15}\end{aligned}$$

To get the probability for the second situation (that it does not rain tomorrow, but it does rain the day after tomorrow) we have to multiply the probabilies along the second red line.

$$
\begin{aligned}
& P(\text{not rain tomorrow and rain day after tomorrow}) \\
=& \frac{4}{5} \times \frac{1}{6} \\
=& \frac{2}{15}
\end{aligned}
$$

Therefore the total probability that it will rain the day after tomorrow is the sum of the probabilities along the two red paths, namely

$$\frac{1}{15} + \frac{2}{15} = \frac{1}{5}$$

Worked example 11: Drawing a tree diagram

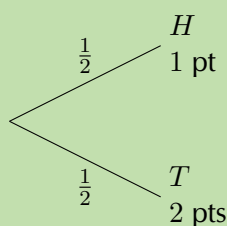
QUESTION

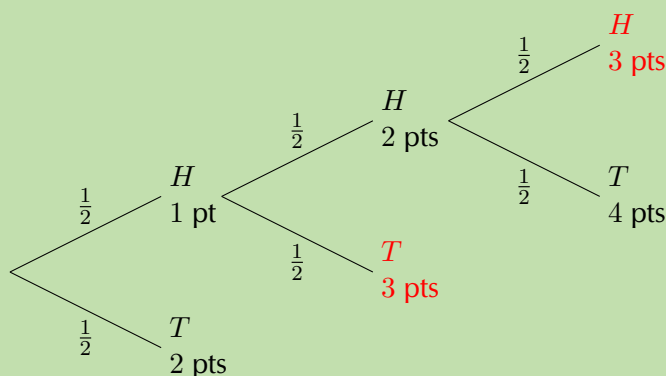
You play the following game. You flip a coin. If it comes up tails, you get 2 points and your turn ends. If it comes up heads, you get only 1 point, but you can flip the coin again. If you flip the coin multiple times in one turn, you add up the points. You can flip the coin at most 3 times in one turn. What is the probability that you will get exactly 3 points in one turn? Draw a tree diagram to visualise the different possibilities.

SOLUTION

Step 1: Write down the events and their symbols

Each coin toss has on of two possible outcomes, namely heads (H) and tails (T). Each outcome has a probability of $\frac{1}{2}$. We are asked to count the number of points, so we will also indicate how many points we have for each outcome.

Step 2: Draw the first level of the tree diagram

This tree diagram shows the possible outcomes after 1 flip of the coin. Remember that we can have up to 3 flips, so the diagram is not complete yet. If the coin comes up heads, we flip the coin again. If the coin comes up tails, we stop.

Step 3: Draw the second and third level of the tree diagram

In this tree diagram you can see that we add up the points we get with each coin flip. After three coin flips, the game is over.

Step 4: Find the relevant outcomes and compute the probability

We are interested in getting exactly 3 points during the game. To find these outcomes we look only at the tips of the tree. We end with exactly 3 points when the coin flips are

- $(H;T)$ with probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$;
- $(H;H;H)$ with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Notice that we compute the probability of an outcome by multiplying all the probabilities along the path from the start of the tree to the tip where the outcome is. We add the above two probabilites to obtain the final probability of getting exactly 3 points as $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$.

Worked example 12: Drawing a tree diagram

QUESTION

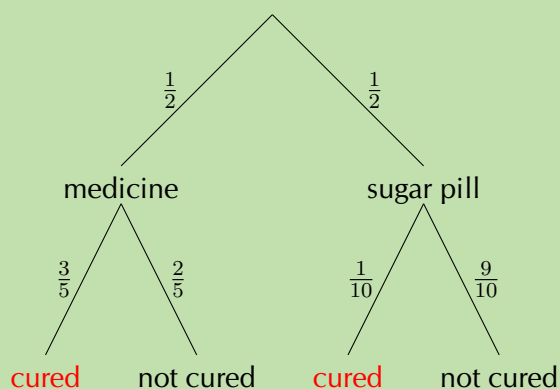
A person takes part in a medical trial that tests the effect of a medicine on a disease. Half the people are given medicine and the other half are given a sugar pill, which has no effect on the disease. The medicine has a 60% chance of curing someone. But, people who do not get the medicine still have a 10% chance of getting well. There are 50 people in the trial and they all have the disease. Talwar takes part in the trial, but we do not know whether he got the medicine or the sugar pill. Draw a tree diagram of all the possible cases. What is the probability that Talwar gets cured?

SOLUTION

Step 1: Summarise the information in the problem

There are two uncertain events in this problem. Each person either receives medicine (probability $\frac{1}{2}$) or a sugar pill (probability $\frac{1}{2}$). Each person also gets cured (probability

$\frac{3}{5}$ with medicine and $\frac{1}{10}$ without) or stays ill (probability $\frac{2}{5}$ with medicine and $\frac{9}{10}$ without).

Step 2: Draw the tree diagram

In the first level of the tree diagram we show that Talwar either gets the medicine or the sugar pill. The second level of the tree diagram shows whether Talwar is cured or not, depending on which one of the pills he got.

Step 3: Compute the required probability

We multiply the probabilites along each path in the tree diagram that leads to Talwer being cured:

$$\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$
$$\frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$$

We then add these probabilites to get the final answer. The probability that Talwar is cured is $\frac{7}{20}$.

Exercise 10 – 5: Tree diagrams

1. You roll a die twice and add up the dots to get a score. Draw a tree diagram to represent this experiment. What is the probability that your score is a multiple of 5?
2. What is the probability of throwing at least one five in four rolls of a regular 6-sided die? Hint: do not show all possible outcomes of each roll of the die. We are interested in whether the outcome is 5 or not 5 only.
3. You flip one coin 4 times.
   a) What is the probability of getting exactly 3 heads?
   b) What is the probability of getting at least 3 heads?

4. You flip 4 different coins at the same time.
    a) What is the probability of getting exactly 3 heads?
    b) What is the probability of getting at least 3 heads?

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1. 23BK 2. 23BM 3. 23BN 4. 23BP

10.5 Contingency tables EMBJX

A contingency table is another tool for keeping a record of the counts or percentages in a probability problem. Contingency tables are especially helpful for figuring out whether events are dependent or independent.

We will be studying two-way contingency tables, where we count the number of outcomes for 2 events and their complements, making 4 events in total. A two-way contingency table always shows the counts for the 4 possible combinations of events, as well as the totals for each event and its complement. We can use a contingency table to compute the probabilities of various events by computing the ratios between counts, and to determine whether the events are dependent or independent. The example below shows a two-way contingency table, representing the outcome of a medical study.

Worked example 13: Contingency tables

QUESTION

A medical trial into the effectiveness of a new medication was carried out. 120 females and 90 males took part in the trial. Out of those people, 50 females and 30 males responded positively to the medication. Given below is a contingency table with the given information filled in.

	Female	Male	Totals
Positive	50	30	
Negative			
Totals	120	90	

1. What is the probability that the medicine gives a positive result for females?
2. What is the probability that the medicine gives a negative result for males?
3. Was the medication's success independent of gender? Explain.

SOLUTION

Step 1: Complete the contingency table

The best place to start is always to complete the contingency table. Because the each column has to sum up to its total, we can work out the number of females and males who responded negatively to the medication. Then we can add each row to get the totals on the right hand side of the table.

	Female	Male	Totals
Positive	50	30	80
Negative	70	60	130
Totals	120	90	210

Step 2: Compute the required probabilities

The way the first question is phrased, we need to determine the probability that a person responds positively if she is female. This means that we do not include males in this calculation. So, the probability that the medicine gives a positive result for females is the ratio between the number of females who got a positive response and the total number of females.

$$\begin{aligned} P(\text{positive if female}) &= \frac{n(\text{positive and female})}{n(\text{female})} \\ &= \frac{50}{120} \\ &= \frac{5}{12} \end{aligned}$$

Similarly, the probability that the medicine gives a negative result for males is:

$$\begin{aligned} P(\text{negative if male}) &= \frac{n(\text{negative and male})}{n(\text{male})} \\ &= \frac{60}{90} \\ &= \frac{2}{3} \end{aligned}$$

Step 3: Independence

We need to determine whether the effect of the medicine and the gender of a participant are dependent or independent. According to the definition, two events are independent if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

We will look at the events that a participant is female and that the participant responded positively to the trial.

$$\begin{aligned} P(\text{female}) &= \frac{n(\text{female})}{n(\text{total trials})} \\ &= \frac{120}{210} \\ &= \frac{4}{7} \end{aligned}$$

$$
\begin{aligned}
P(\text{positive}) &= \frac{n(\text{positive})}{n(\text{total trials})} \\
&= \frac{80}{210} \\
&= \frac{8}{21}
\end{aligned}
$$

$$
\begin{aligned}
P(\text{female and positive}) &= \frac{n(\text{female and positive})}{n(\text{total trials})} \\
&= \frac{50}{210} \\
&= \frac{5}{21}
\end{aligned}
$$

From these probabilities we can see that

$$P(\text{female and positive}) \neq P(\text{female}) \times P(\text{positive})$$

and therefore the gender of a participant and the outcome of a trial are dependent events.

Worked example 14: Contingency tables

QUESTION

Use the contingency table below to answer the following questions.

	Grade 11	Grade 12	Totals
Has cellphone	59	50	109
No cellphone	6	3	9
Totals	65	53	118

1. What is the probability that a learner from Grade 11 has a cellphone?
2. What is the probability that a learner who does not have a cellphone is from Grade 11.
3. Are the grade of a learner and whether he has a cellphone or not independent events? Explain your answer.

SOLUTION

1. There are 65 learners in Grade 11 and 59 of them have a cellphone. Therefore the probability that a learner from Grade 11 has a cellphone is $\frac{59}{65}$.
2. There are 9 learners who do not have a cellphone and 6 of them are in Grade 11. Therefore the probability that a learner who does not have a cellphone is from from Grade 11 is $\frac{6}{9} = \frac{2}{3}$.

3. To test for independence, we will consider whether a learner is in Grade 11 and whether a learner has a cellphone. The probability that a learner is in Grade 11 is $\frac{65}{118}$. The probability that a learner has a cellphone is $\frac{109}{118}$. The probability that a learner is in Grade 11 and has a cellphone is $\frac{59}{118} = \frac{1}{2}$. Since $\frac{1}{2} \neq \frac{65}{118} \times \frac{109}{118}$ the grade of a learner and whether he has a cellphone are dependent.

Exercise 10 – 6: Contingency tables

1. Use the contingency table below to answer the following questions.

	Brown eyes	Not brown eyes	Totals
Black hair	50	30	80
Red hair	70	80	150
Totals	120	110	230

   a) What is the probability that someone with black hair has brown eyes?
   b) What is the probability that someone has black hair?
   c) What is the probability that someone has brown eyes?
   d) Are having black hair and having brown eyes dependent or independent events?

2. Given the following contingency table, identify the events and determine whether they are dependent or independent.

	Location A	Location B	Totals
Buses left late	15	40	55
Buses left on time	25	20	45
Totals	40	60	100

3. You are given the following information.

   - Events A and B are independent.
   - $P(\text{not } A) = 0{,}3$.
   - $P(B) = 0{,}4$.

   Complete the contingency table below.

	A	not A	Totals
B			
not B			
Totals			50

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1. 23BQ 2. 23BR 3. 23BS

10.6 Summary

EMBJY

See presentation: 23BT at www.everythingmaths.co.za

- Terminology:
  - **Outcome**: a single observation of an experiment.
  - **Sample space** of an experiment: the set of all possible outcomes of the experiment.
  - **Event**: a set of outcomes of an experiment.
  - **Probability** of an event: a real number between 0 and 1 that describes how likely it is that the event will occur.
  - **Relative frequency** of an event: the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.
  - **Union** of events: the set of all outcomes that occur in at least one of the events, written as "A or B".
  - **Intersection** of events: the set of all outcomes that occur in all of the events, written as "A and B".
  - **Mutually exclusive events**: events with no outcomes in common, that is $(A \text{ and } B) = \emptyset$.
  - **Complementary events**: two mutually exclusive events that together contain all the outcomes in the sample space. We write the complement as "not A".
  - **Independent events**: two events where knowing the outcome of one event does not affect the probability of the other event. Events are independent if and only if $P(A \text{ and } B) = P(A) \times P(B)$.
- Identities:
  - The **addition rule**: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  - The **addition rule for 2 mutually exclusive events**: $P(A \text{ or } B) = P(A) + P(B)$
  - The **complementary rule**: $P(\text{not } A) = 1 - P(A)$
- A **Venn diagram** is a visual tool used to show how events overlap. Each region in a Venn diagram represents an event and could contain either the outcomes in the event, the number of outcomes in the event or the probability of the event.
- A **tree diagram** is a visual tool that helps with computing probabilities for dependent events. The outcomes of each event are shown along with the probability of each outcome. For each event that depends on a previous event, we go one level deeper into the tree. To compute the probability of some combination of outcomes, we
  - find all the paths that contain the outcome of interest;
  - multiply the probabilities along each path;
  - add the probabilities between different paths.
- A **2-way contingency table** is a tool for organising data, especially when we want to determine whether two events, each with only two outcomes, are dependent or independent. The counts for each possible combination of outcomes are entered into the table, along with the totals of each row and column.

Exercise 10 – 7: End of chapter exercises

1. Jane invested in the stock market. The probability that she will not lose all her money is 0,32. What is the probability that she will lose all her money? Explain.
2. If D and F are mutually exclusive events, with $P(\text{not } D) = 0{,}3$ and $P(D \text{ or } F) = 0{,}94$, find $P(F)$.
3. A car sales person has pink, lime-green and purple models of car A and purple, orange and multicolour models of car B. One dark night a thief steals a car.
   a) What is the experiment and sample space?
   b) What is the probability of stealing either a model of A or a model of B?
   c) What is the probability of stealing both a model of A and a model of B?
4. The probability of event X is 0,43 and the probability of event Y is 0,24. The probability of both occurring together is 0,10. What is the probability that X or Y will occur?
5. $P(H) = 0{,}62$; $P(J) = 0{,}39$ and $P(H \text{ and } J) = 0{,}31$. Calculate:
   a) $P(H')$
   b) $P(H \text{ or } J)$
   c) $P(H' \text{ or } J')$
   d) $P(H' \text{ or } J)$
   e) $P(H' \text{ and } J')$
6. The last ten letters of the alphabet are placed in a hat and people are asked to pick one of them. Event D is picking a vowel, event E is picking a consonant and event F is picking one of the last four letters. Draw a Venn diagram showing the outcomes in the sample space and the different events. Then calculate the following probabilities:
   a) $P(\text{not } F)$
   b) $P(F \text{ or } D)$
   c) $P(\text{neither } E \text{ nor } F)$
   d) $P(D \text{ and } E)$
   e) $P(E \text{ and } F)$
   f) $P(E \text{ and } D')$
7. Thobeka compares three neighbourhoods (we'll call them A, B and C) to see where the best place is to live. She interviews 80 people and asks them whether they like each of the neighbourhoods, or not.
   - 40 people like neighbourhood A.
   - 35 people like neighbourhood B.
   - 40 people like neighbourhood C.
   - 21 people like both neighbourhoods A and C.
   - 18 people like both neighbourhoods B and C.
   - 68 people like at least one neighbourhood.
   - 7 people like all three neighbourhoods.

a) Use this information to draw a Venn diagram.

b) How many people like none of the neighbourhoods?

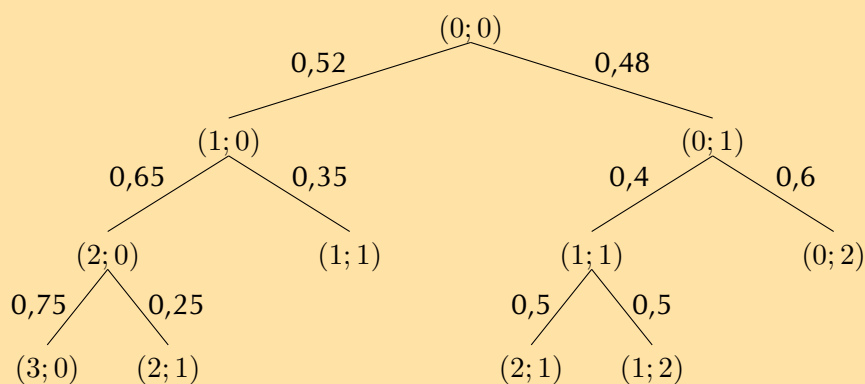
c) How many people like neighbourhoods A and B, but not C?

d) What is the probability that a randomly chosen person from the survey likes at least one of the neighbourhoods?

8. Let G and H be two events in a sample space. Suppose that $P(G) = 0{,}4$; $P(H) = h$; and $P(G \text{ or } H) = 0{,}7$.

   a) For what value of h are G and H mutually exclusive?

   b) For what value of h are G and H independent?

9. The following tree diagram represents points scored by two teams in a soccer game. At each level in the tree, the points are shown as (points for Team 1; points for Team 2).

   Use this diagram to determine the probability that:

   a) Team 1 will win

   b) The game will be a draw

   c) The game will end with an even number of total points

10. A bag contains 10 orange balls and 7 black balls. You draw 3 balls from the bag **without replacement**. What is the probability that you will end up with exactly 2 orange balls? Represent this experiment using a tree diagram.

11. Complete the following contingency table and determine whether the events are dependent or independent.

	Durban	Bloemfontein	Totals
Liked living there	130	30	
Did not like living there	140		340
Totals		230	500

12. Summarise the following information about a medical trial with 2 types of multi-vitamin in a contingency table and determine whether the events are dependent or independent.

   - 960 people took part in the medical trial.
   - 540 people used multivitamin A for a month and 400 of those people showed an improvement in their health.

- 300 people showed an improvement in health when using multivitamin B for a month.

If the events are independent, it means that the two multivitamins have the same effect on people. If the events are dependent, it means that one multivitamin is better than the other. Which multivitamin is better than the other, or are the both equally effective?

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1. 23BV	2. 23BW	3. 23BX	4. 23BY	5. 23BZ	6. 23C2
7. 23C3	8. 23C4	9. 23C5	10. 23C6	11. 23C7	12. 23C8

CHAPTER

Statistics

11.1 Revision EMBJZ

Measures of central tendency EMBK2

The mean and median of a data set both give an indication where the centre of the data distribution is located. The **mean**, or average, is calculated as

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

where the x_i are the data and n is the number of data. We read $\overline{x}$ as "x bar".

The **median** is the middle value of an ordered data set. To find the median, we first sort the data and then pick out the value in the middle of the sorted list. If the middle is in between two values, the median is the average of those two values.

See video: 23C9 at www.everythingmaths.co.za

Worked example 1: Computing measures of central tendency

QUESTION

Compute the mean and median of the following data set:

72,5 ; 92,6 ; 15,6 ; 53,0 ; 86,4 ; 89,9 ; 90,9 ; 21,7 ; 46,0 ; 4,1 ; 51,7 ; 2,2

SOLUTION

Step 1: Compute the mean

Using the formula for the mean, we first compute the sum of the values and then divide by the number of values.

$$\begin{aligned}\overline{x} &= \frac{626{,}6}{12} \\ &\approx 52{,}22\end{aligned}$$

Step 2: Compute the median

To find the median, we first have to sort the data:

2,2 ; 4,1 ; 15,6 ; 21,7 ; 46,0 ; 51,7 ; 53,0 ; 72,5 ; 86,4 ; 89,9 ; 90,9 ; 92,6

Since there are an even number of values, the median will lie between two values. In this case, the two values in the middle are 51,7 and 53,0. Therefore the median is 52,35.

Measures of dispersion EMBK3

Measures of dispersion tell us how spread out a data set is. If a measure of dispersion is small, the data are clustered in a small region. If a measure of dispersion is large, the data are spread out over a large region.

The **range** is the difference between the maximum and minimum values in the data set.

The **inter-quartile range** is the difference between the first and third quartiles of the data set. The quartiles are computed in a similar way to the median. The median is halfway into the ordered data set and is sometimes also called the second quartile. The first quartile is one quarter of the way into the ordered data set; whereas the third quartile is three quarters of the way into the ordered data set.

See video: 23CB at www.everythingmaths.co.za

Worked example 2: Range and inter-quartile range

QUESTION

Determine the range and the inter-quartile range of the following data set.

$$14\ ;\ 17\ ;\ 45\ ;\ 20\ ;\ 19\ ;\ 36\ ;\ 7\ ;\ 30\ ;\ 8$$

SOLUTION

Step 1: Sort the values in the data set

To determine the range we need to find the minimum and maximum values in the data set. To determine the inter-quartile range we need to compute the first and third quartiles of the data set. For both of these requirements, it is easier to order the data set first.

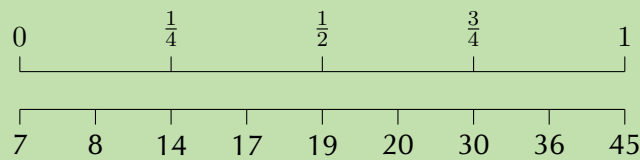
The sorted data set is

$$7\ ;\ 8\ ;\ 14\ ;\ 17\ ;\ 19\ ;\ 20\ ;\ 30\ ;\ 36\ ;\ 45$$

Step 2: Find the minimum, maximum and range

The minimum value is the first value in the ordered data set, namely 7. The maximum is the last value in the ordered data set, namely 45. The range is the difference between the minimum and maximum: $45 - 7 = 38$.

Step 3: Find the quartiles and inter-quartile range

The diagram below shows how we find the quartiles one quarter, one half and three quarters of the way into the ordered list of values.

From this diagram we can see that the first quartile is at a value of 14, the second quartile (median) is at a value of 19 and the third quartile is at a value of 30.

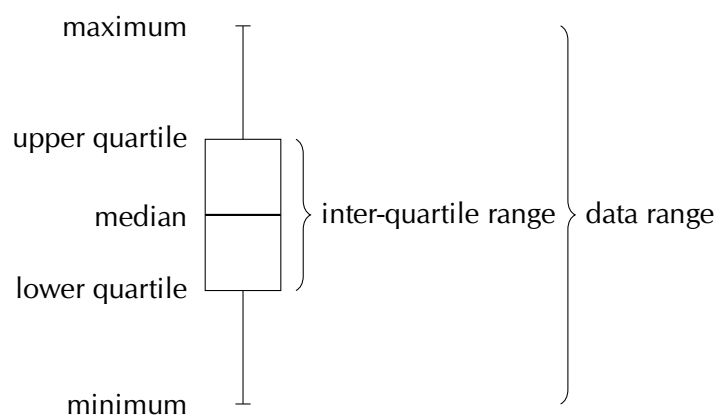
The inter-quartile range is the difference between the first and third quartiles. The first quartile is 14 and the third quartile is 30. Therefore the inter-quartile range is $30 - 14 = 16$.

Five number summary

EMBK4

The **five number summary** combines a measure of central tendency, namely the median, with measures of dispersion, namely the range and the inter-quartile range. This gives a good overview of the overall data distribution. More precisely, the five number summary is written in the following order:

- minimum;
- first quartile;
- median;
- third quartile;
- maximum.

The five number summary is often presented visually using a **box and whisker diagram**. A box and whisker diagram is shown below, with the positions of the five relevant numbers labelled. Note that this diagram is drawn vertically, but that it may also be drawn horizontally.

See video: 23CC at www.everythingmaths.co.za

Worked example 3: Five number summary

QUESTION

Draw a box and whisker diagram for the following data set:

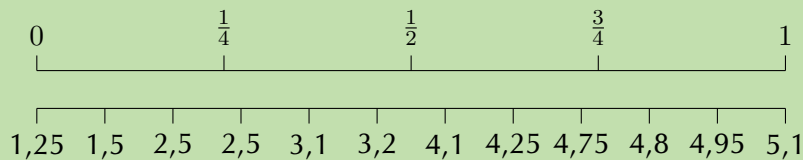
$$1{,}25\ ;\ 1{,}5\ ;\ 2{,}5\ ;\ 2{,}5\ ;\ 3{,}1\ ;\ 3{,}2\ ;\ 4{,}1\ ;\ 4{,}25\ ;\ 4{,}75\ ;\ 4{,}8\ ;\ 4{,}95\ ;\ 5{,}1$$

SOLUTION

Step 1: Determine the minimum and maximum

Since the data set is already ordered, we can read off the minimum as the first value (1,25) and the maximum as the last value (5,1).

Step 2: Determine the quartiles

There are 12 values in the data set.

Using the figure above we can see that the median is between the sixth and seventh values, making it.

$$\frac{3{,}2+4{,}1}{2}=3{,}65$$

The first quartile lies between the third and fourth values, making it

$$Q_1=\frac{2{,}5+2{,}5}{2}=2{,}5$$

The third quartile lies between the ninth and tenth values, making it

$$Q_3=\frac{4{,}75+4{,}8}{2}=4{,}775$$

Step 3: Draw the box and whisker diagram

We now have the five number summary as (1,25; 2,5; 3,65; 4,775; 5,1). The box and whisker diagram representing the five number summary is given below.

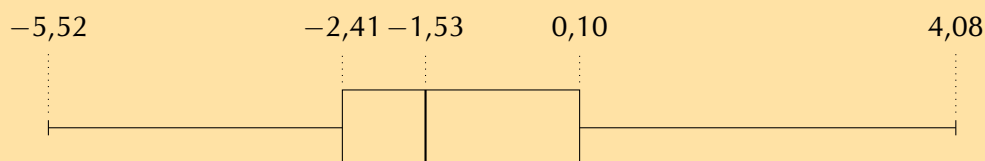
Exercise 11 – 1: Revision

1. For each of the following data sets, compute the mean and all the quartiles. Round your answers to one decimal place.

   a) $-3{,}4$; $-3{,}1$; $-6{,}1$; $-1{,}5$; $-7{,}8$; $-3{,}4$; $-2{,}7$; $-6{,}2$

   b) -6 ; -99 ; 90 ; 81 ; 13 ; -85 ; -60 ; 65 ; -49

   c) 7 ; 45 ; 11 ; 3 ; 9 ; 35 ; 31 ; 7 ; 16 ; 40 ; 12 ; 6

2. Use the following box and whisker diagram to determine the range and interquartile range of the data.

3. Draw the box and whisker diagram for the following data.

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1a. 23CD 1b. 23CF 1c. 23CG 2. 23CH 3. 23CJ

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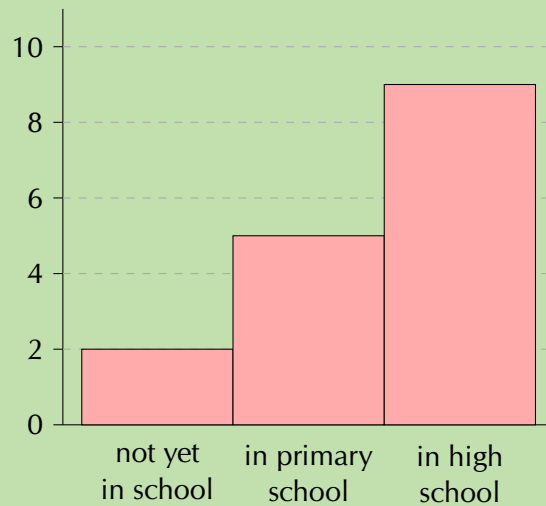
11.2 Histograms EMBK5

A histogram is a graphical representation of how many times different, mutually exclusive events are observed in an experiment. To interpret a histogram, we find the events on the x-axis and the counts on the y-axis. Each event has a rectangle that shows what its count (or frequency) is.

See video: 23CK at www.everythingmaths.co.za

Worked example 4: Reading histograms

QUESTION

Use the following histogram to determine the events that were recorded and the relative frequency of each event. Summarise your answer in a table.

SOLUTION

Step 1: Determine the events

The events are shown on the x-axis. In this example we have "not yet in school", "in primary school" and "in high school".

Step 2: Read off the count for each event

The counts are shown on the y-axis and the height of each rectangle shows the frequency for each event.

- not yet in school: 2
- in primary school: 5
- in high school: 9

Step 3: Calculate relative frequency

The relative frequency of an event in an experiment is the number of times that the event occurred divided by the total number of times that the experiment was completed. In this example we add up the frequencies for all the events to get a total frequency of 16. Therefore the relative frequencies are:

- not yet in school: $\frac{2}{16} = \frac{1}{8}$
- in primary school: $\frac{5}{16}$
- in high school: $\frac{9}{16}$

Step 4: Summarise

Event	Count	Relative frequency
not yet in school	2	$\frac{1}{8}$
in primary school	5	$\frac{5}{16}$
in high school	9	$\frac{9}{16}$

To draw a histogram of a data set containing numbers, the numbers first have to be grouped. Each group is defined by an interval. We then count how many times numbers from each group appear in the data set and draw a histogram using the counts.

Worked example 5: Draw a histogram

QUESTION

The following data represent the heights of 16 adults in centimetres.

162 ; 168 ; 177 ; 147 ; 189 ; 171 ; 173 ; 168
178 ; 184 ; 165 ; 173 ; 179 ; 166 ; 168 ; 165

Divide the data into 5 equal length intervals between 140 cm and 190 cm and draw a histogram.

SOLUTION

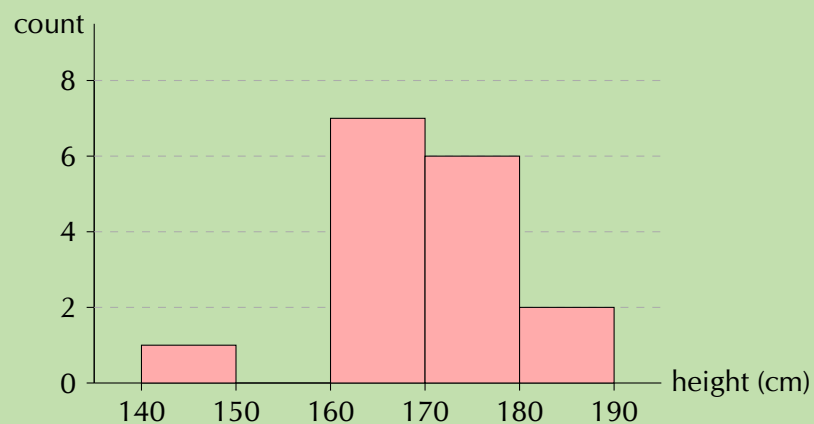
Step 1: Determine intervals

To have 5 intervals of the same length between 140 and 190, we need and interval length of 10. Therefore the intervals are $(140; 150]$; $(150; 160]$; $(160; 170]$; $(170; 180]$; and $(180; 190]$.

Step 2: Count data

The following table summarises the number of data values in each of the intervals.

Interval	$(140; 150]$	$(150; 160]$	$(160; 170]$	$(170; 180]$	$(180; 190]$
Count	1	0	7	6	2

Step 3: Draw the histogram

A frequency polygon is sometimes used to represent the same information as in a histogram. A frequency polygon is drawn by using line segments to connect the middle of the top of each bar in the histogram. This means that the frequency polygon connects the coordinates at the centre of each interval and the count in each interval.

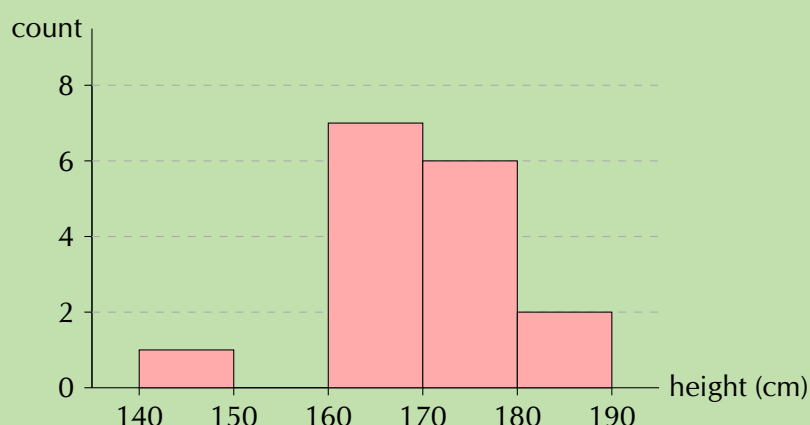
Worked example 6: Drawing a frequency polygon

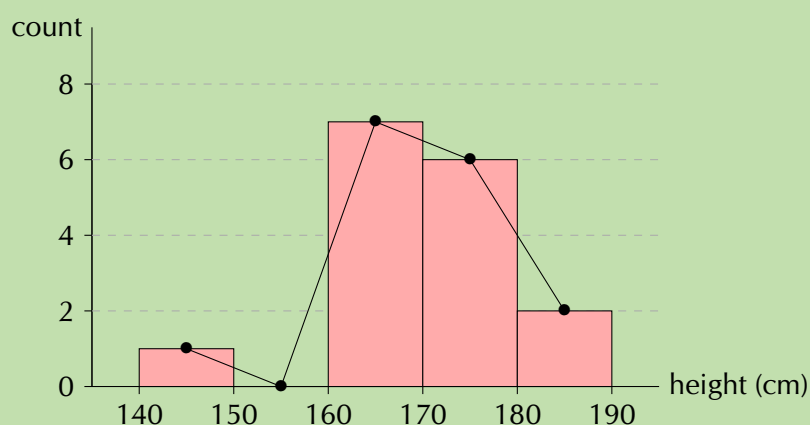
QUESTION

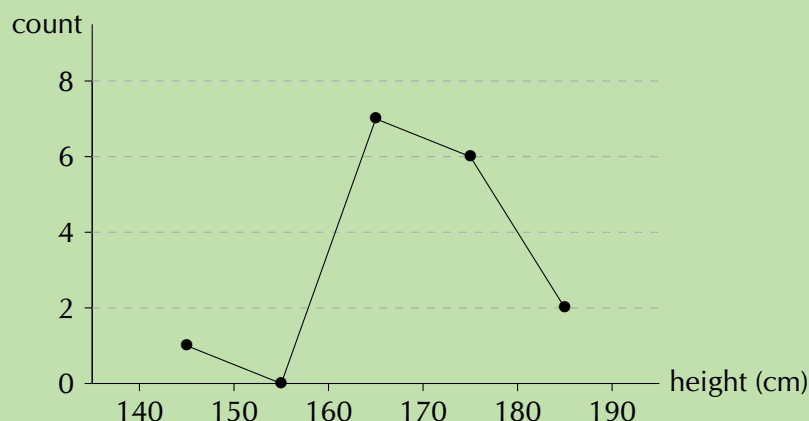
Use the histogram from the previous example to draw a frequency polygon of the same data.

SOLUTION

Step 1: Draw the histogram

We already know that the histogram looks like this:

Step 2: Connect the tops of the rectangles

When we draw line segments between the tops of the rectangles in the histogram, we get the following picture:

Step 3: Draw final frequency polygon

Finally, we remove the histogram to show only the frequency polygon.

Frequency polygons are particularly useful for comparing two data sets. Comparing two histograms would be more difficult since we would have to draw the rectangles of the two data sets on top of each other. Because frequency polygons are just lines, they do not pose the same problem.

Worked example 7: Drawing frequency polygons

QUESTION

Here is another data set of heights, this time of Grade 11 learners.

132 ; 132 ; 156 ; 147 ; 162 ; 168 ; 152 ; 174
141 ; 136 ; 161 ; 148 ; 140 ; 174 ; 174 ; 162

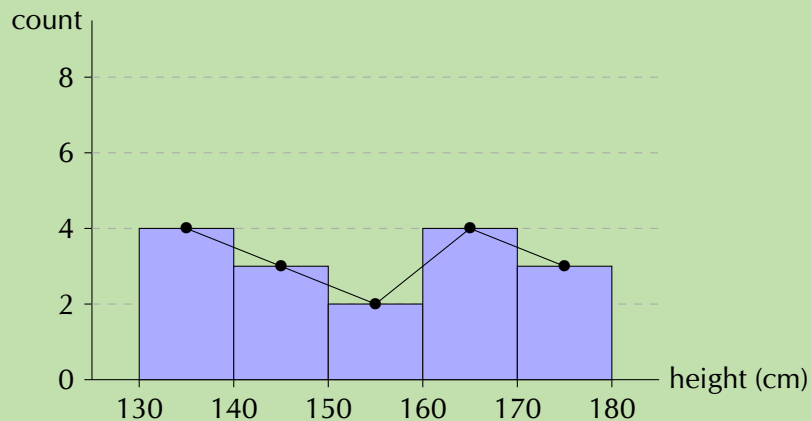
Draw the frequency polygon for this data set using the same interval length as in the previous example. Then compare the two frequency polygons on one graph to see the differences between the distributions.

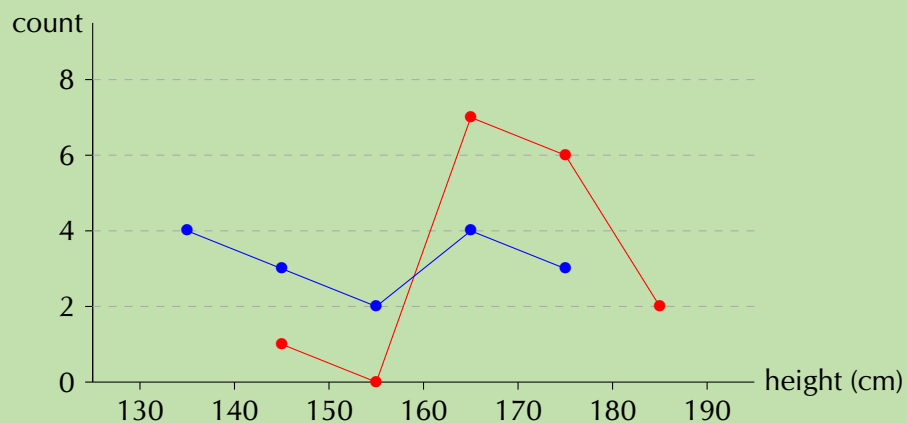
SOLUTION

Step 1: Frequency table

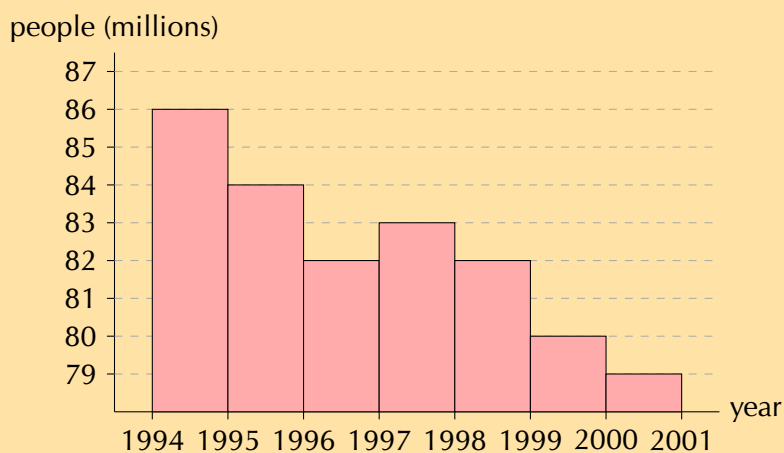
We first create the table of counts for the new data set.

Interval	(130; 140]	(140; 150]	(150; 160]	(160; 170]	(170; 180]
Count	4	3	2	4	3

Step 2: Draw histogram and frequency polygon

Step 3: Compare frequency polygons

We draw the two frequency polygons on the same axes. The red line indicates the distribution over heights for adults and the blue line, for Grade 11 learners.

From this plot we can easily see that the heights for Grade 11 learners are distributed more towards the left (shorter) than adults. The learner heights also seem to be more evenly distributed between 130 and 180 cm, whereas the adult heights are mostly between 160 and 180 cm.

Exercise 11 – 2: Histograms

1. Use the histogram below to answer the following questions. The histogram shows the number of people born around the world each year. The ticks on the x-axis are located at the start of each year.

   a) How many people were born between the beginning of 1994 and the beginning of 1996?

   b) Is the number people in the world population increasing or decreasing? (Ignore the rate at which people are dying for this question.)

   c) How many more people were born in 1994 than in 1997?

2. In a traffic survey, a random sample of 50 motorists were asked the distance (d) they drove to work daily. The results of the survey are shown in the table below. Draw a histogram to represent the data.

d	$0 < d \leq 10$	$10 < d \leq 20$	$20 < d \leq 30$	$30 < d \leq 40$	$40 < d \leq 50$
f	9	19	15	5	4

3. Below is data for the prevalence of HIV in South Africa. HIV prevalence refers to the percentage of people between the ages of 15 and 49 who are infected with HIV.

year	2002	2003	2004	2005	2006	2007	2008	2009
prevalence (%)	17,7	18,0	18,1	18,1	18,1	18,0	17,9	17,9

   Draw a frequency polygon of this data set.

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1. 23CM 2. 23CN 3. 23CP

11.3 Ogives EMBK7

Cumulative histograms, also known as ogives, are graphs that can be used to determine how many data values lie above or below a particular value in a data set. The cumulative frequency is calculated from a frequency table, by adding each frequency to the total of the frequencies of all data values before it in the data set. The last value for the cumulative frequency will always be equal to the total number of data values, since all frequencies will already have been added to the previous total.

An ogive is drawn by

- plotting the beginning of the first interval at a y-value of zero;
- plotting the end of every interval at the y-value equal to the cumulative count for that interval; and
- connecting the points on the plot with straight lines.

In this way, the end of the final interval will always be at the total number of data since we will have added up across all intervals.

Worked example 8: Cumulative frequencies and ogives

QUESTION

Determine the cumulative frequencies of the following grouped data and complete the table below. Use the table to draw an ogive of the data.

Interval	Frequency	Cumulative frequency
$10 < n \leq 20$	5	
$20 < n \leq 30$	7	
$30 < n \leq 40$	12	
$40 < n \leq 50$	10	
$50 < n \leq 60$	6	

SOLUTION

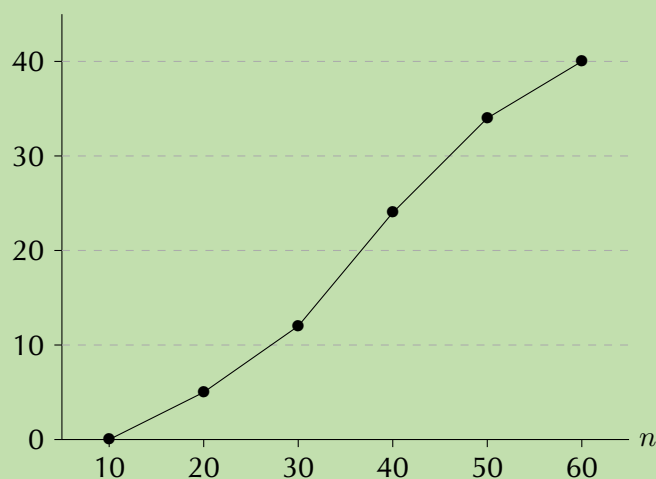
Step 1: Compute cumulative frequencies

To determine the cumulative frequency, we add up the frequencies going down the table. The first cumulative frequency is just the same as the frequency, because we are adding it to zero. The final cumulative frequency is always equal to the sum of all the frequencies. This gives the following table:

Interval	Frequency	Cumulative frequency
$10 < n \leq 20$	5	5
$20 < n \leq 30$	7	12
$30 < n \leq 40$	12	24
$40 < n \leq 50$	10	34
$50 < n \leq 60$	6	40

Step 2: Plot the ogive

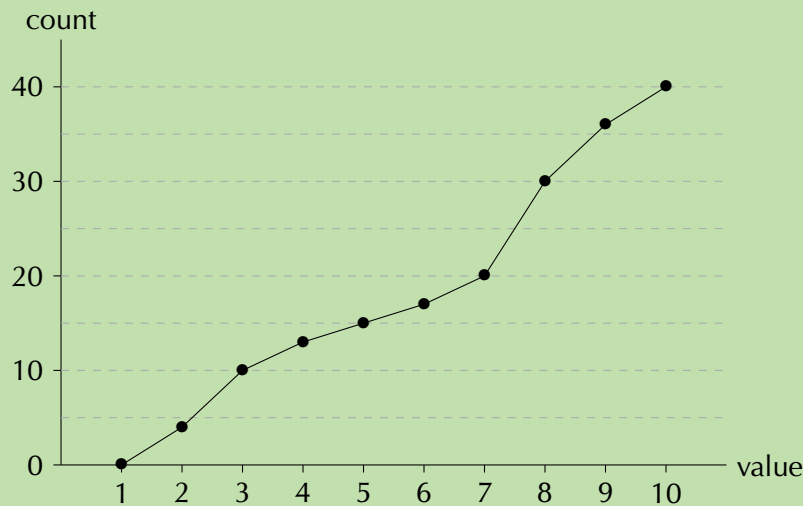
The first coordinate in the plot always starts at a y-value of 0 because we always start from a count of zero. So, the first coordinate is at $(10; 0)$ — at the beginning of the first interval. The second coordinate is at the end of the first interval (which is also the beginning of the second interval) and at the first cumulative count, so $(20; 5)$. The third coordinate is at the end of the second interval and at the second cumulative count, namely $(30; 12)$, and so on.

Computing all the coordinates and connecting them with straight lines gives the following ogive.

Ogives do look similar to frequency polygons, which we saw earlier. The most important difference between them is that an ogive is a plot of **cumulative** values, whereas a frequency polygon is a plot of the values themselves. So, to get from a frequency polygon to an ogive, we would add up the counts as we move from left to right in the graph.

Ogives are useful for determining the median, percentiles and five number summary of data. Remember that the median is simply the value in the middle when we order the data. A quartile is simply a quarter of the way from the beginning or the end of an ordered data set. With an ogive we already know how many data values are above or below a certain point, so it is easy to find the middle or a quarter of the data set.

Worked example 9: Ogives and the five number summary

QUESTION

Use the following ogive to compute the five number summary of the data. Remember that the five number summary consists of the minimum, all the quartiles (including the median) and the maximum.

SOLUTION

Step 1: Find the minimum and maximum

The minimum value in the data set is 1 since this is where the ogive starts on the horizontal axis. The maximum value in the data set is 10 since this is where the ogive stops on the horizontal axis.

Step 2: Find the quartiles

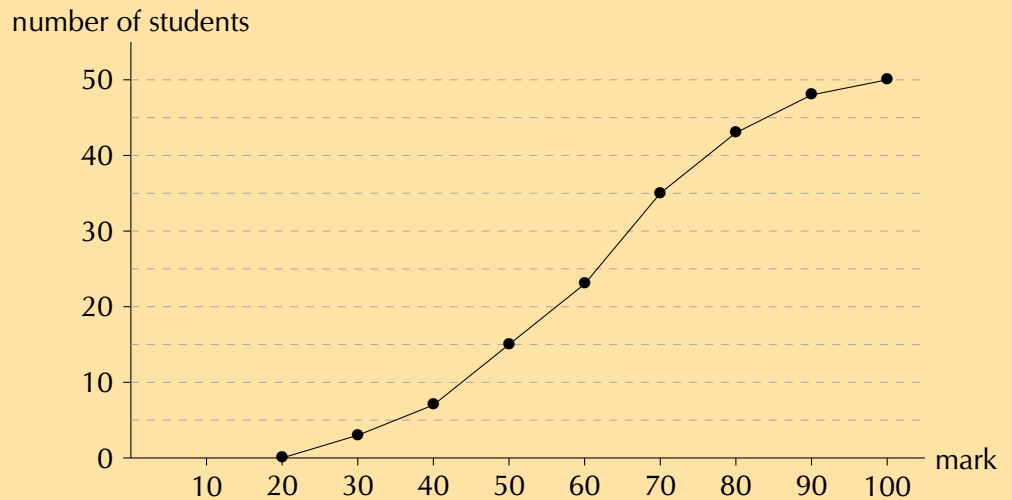
The quartiles are the values that are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ of the way into the ordered data set. Here the counts go up to 40, so we can find the quartiles by looking at the values corresponding to counts of 10, 20 and 30. On the ogive a count of

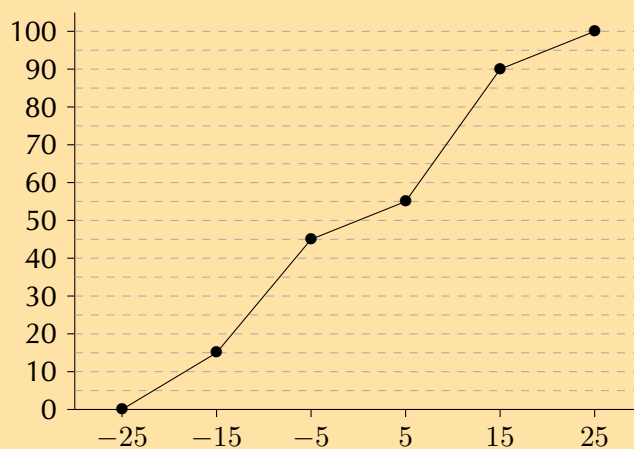
- 10 corresponds to a value of 3 (first quartile);
- 20 corresponds to a value of 7 (second quartile); and
- 30 corresponds to a value of 8 (third quartile).

Step 3: Write down the five number summary

The five number summary is $(1; 3; 7; 8; 10)$. The box-and-whisker plot of this data set is given below.

Exercise 11 – 3: Ogives

1. Use the ogive to answer the questions below. Note that marks are given as a percentage.

   a) How many students got between 50% and 70%?
   b) How many students got at least 70%?
   c) Compute the average mark for this class, rounded to the nearest integer.

2. Draw the histogram corresponding to this ogive.

3. The following data set lists the ages of 24 people.

   2; 5; 1; 76; 34; 23; 65; 22; 63; 45; 53; 38

   4; 28; 5; 73; 79; 17; 15; 5; 34; 37; 45; 56

   Use the data to answer the following questions.

   a) Using an interval width of 8 construct a cumulative frequency plot.
   b) How many are below 30?
   c) How many are below 60?
   d) Giving an explanation state below what value the bottom 50% of the ages fall.
   e) Below what value do the bottom 40% fall?
   f) Construct a frequency polygon.

4. The weights of bags of sand in grams is given below (rounded to the nearest tenth):

   50,1; 40,4; 48,5; 29,4; 50,2; 55,3; 58,1; 35,3; 54,2; 43,5

   60,1; 43,9; 45,3; 49,2; 36,6; 31,5; 63,1; 49,3; 43,4; 54,1

   a) Decide on an interval width and state what you observe about your choice.
   b) Give your lowest interval.
   c) Give your highest interval.
   d) Construct a cumulative frequency graph and a frequency polygon.
   e) Below what value do 53% of the cases fall?
   f) Below what value of 60% of the cases fall?

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1. 23CQ 2. 23CR 3. 23CS 4. 23CT

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11.4 Variance and standard deviation EMBK8

Measures of central tendency (mean, median and mode) provide information on the data values at the centre of the data set. Measures of dispersion (quartiles, percentiles, ranges) provide information on the spread of the data around the centre. In this section we will look at two more measures of dispersion called the **variance** and the **standard deviation**.

See video: 23CV at www.everythingmaths.co.za

Variance EMBK9

DEFINITION: *Variance*

Let a population consist of n elements, $\{x_1; x_2; \ldots; x_n\}$. Write the mean of the data as $\overline{x}$.
The variance of the data is the average squared distance between the mean and each data value.

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

NOTE:
The variance is written as σ^2. It might seem strange that it is written in squared form, but you will see why soon when we discuss the standard deviation.

The variance has the following properties.

- It is never negative since every term in the variance sum is squared and therefore either positive or zero.
- It has squared units. For example, the variance of a set of heights measured in centimetres will be given in centimeters squared. Since the population variance is squared, it is not directly comparable with the mean or the data themselves. In the next section we will describe a different measure of dispersion, the standard deviation, which has the same units as the data.

Worked example 10: Variance

QUESTION

You flip a coin 100 times and it lands on heads 44 times. You then use the same coin and do another 100 flips. This time in lands on heads 49 times. You repeat this experiment a total of 10 times and get the following results for the number of heads.

$$\{44; 49; 52; 62; 53; 48; 54; 49; 46; 51\}$$

Compute the mean and variance of this data set.

SOLUTION

Step 1: Compute the mean

The formula for the mean is

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

In this case, we sum the data and divide by 10 to get $\overline{x} = 50{,}8$.

Step 2: Compute the variance

The formula for the variance is

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

We first subtract the mean from each datum and then square the result.

x_i	44	49	52	62	53	48	54	49	46	51
$x_i - \overline{x}$	−6,8	−1,8	1,2	11,2	2,2	−2,8	3,2	−1,8	−4,8	0,2
$(x_i - \overline{x})^2$	46,24	3,24	1,44	125,44	4,84	7,84	10,24	3,24	23,04	0,04

The variance is the sum of the last row in this table divided by 10, so $\sigma^2 = 22{,}56$.

Standard deviation EMBKB

Since the variance is a squared quantity, it cannot be directly compared to the data values or the mean value of a data set. It is therefore more useful to have a quantity which is the square root of the variance. This quantity is known as the standard deviation.

DEFINITION: *Standard deviation*

Let a population consist of n elements, $\{x_1; x_2; \ldots; x_n\}$, with a mean of $\overline{x}$. The standard deviation of the data is

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$$

In statistics, the standard deviation is a very common measure of dispersion. Standard deviation measures how spread out the values in a data set are around the mean. More precisely, it is a measure of the average distance between the values of the data in the set and the mean. If the data values are all similar, then the standard deviation will be low (closer to zero). If the data values are highly variable, then the standard variation is high (further from zero).

The standard deviation is always a positive number and is always measured in the same units as the original data. For example, if the data are distance measurements in kilogrammes, the standard deviation will also be measured in kilogrammes.

The mean and the standard deviation of a set of data are usually reported together. In a certain sense, the standard deviation is a natural measure of dispersion if the centre of the data is taken as the mean.

Investigation: Tabulating results

It is often useful to set your data out in a table so that you can apply the formulae easily. Complete the table below to calculate the standard deviation of $\{57; 53; 58; 65; 48; 50; 66; 51\}$.

- Firstly, remember to calculate the mean, $\overline{x}$.
- Complete the following table.

index: i	datum: x_i	deviation: $x_i - \overline{x}$	deviation squared: $(x_i - \overline{x})^2$
1	57		
2	53		
3	58		
4	65		
5	48		
6	50		
7	66		
8	51		
	$\sum x_i = \ldots$	$\sum(x_i - \overline{x}) = \ldots$	$\sum(x_i - \overline{x})^2 = \ldots$

- The sum of the deviations is always zero. Why is this? Find out.
- Calculate the variance using the completed table.
- Then calculate the standard deviation.

Worked example 11: Variance and standard deviation

QUESTION

What is the variance and standard deviation of the possibilities associated with rolling a fair die?

SOLUTION

Step 1: Determine all the possible outcomes

When rolling a fair die, the sample space consists of 6 outcomes. The data set is therefore $x = \{1; 2; 3; 4; 5; 6\}$ and $n = 6$.

Step 2: Calculate the mean

The mean is:

$$\begin{aligned}\overline{x} &= \frac{1}{6}(1+2+3+4+5+6)\\ &= 3{,}5\end{aligned}$$

Step 3: Calculate the variance

The variance is:

$$\begin{aligned}\sigma^2 &= \frac{\sum(x-\overline{x})^2}{n}\\ &= \frac{1}{6}(6{,}25+2{,}25+0{,}25+0{,}25+2{,}25+6{,}25)\\ &= 2{,}917\end{aligned}$$

Step 4: Calculate the standard deviation

The standard deviation is:

$$\begin{aligned}\sigma &= \sqrt{2{,}917}\\ &= 1{,}708\end{aligned}$$

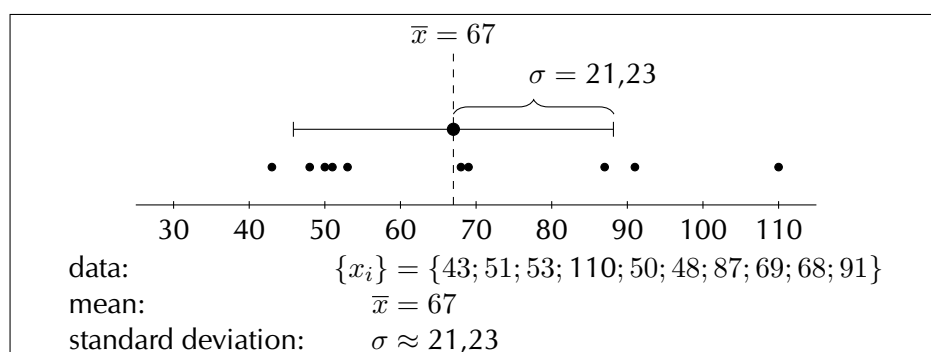
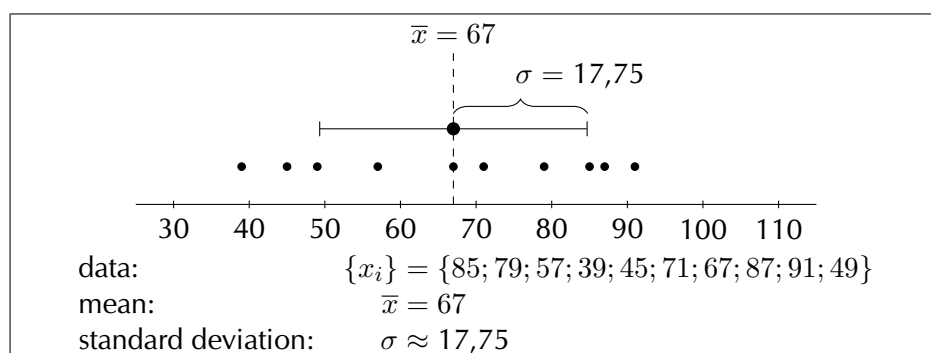
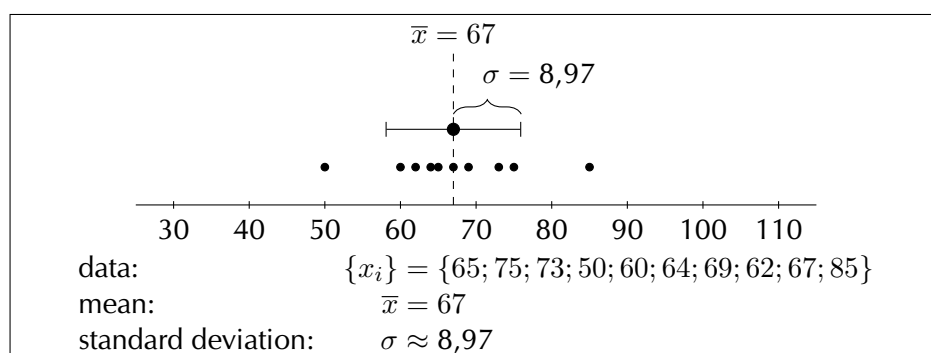
See video: 23CW at www.everythingmaths.co.za

Interpretation and application EMBKC

A large standard deviation indicates that the data values are far from the mean and a small standard deviation indicates that they are clustered closely around the mean.

For example, consider the following three data sets:

$$\{65; 75; 73; 50; 60; 64; 69; 62; 67; 85\}$$
$$\{85; 79; 57; 39; 45; 71; 67; 87; 91; 49\}$$
$$\{43; 51; 53; 110; 50; 48; 87; 69; 68; 91\}$$

Each of these data sets has the same mean, namely 67. However, they have different standard deviations, namely 8,97, 17,75 and 21,23. The following figures show plots of the data sets with the mean and standard deviation indicated on each. You can see how the standard deviation is larger when the data are more spread out.

The standard deviation may also be thought of as a measure of uncertainty. In the physical sciences, for example, the reported standard deviation of a group of repeated measurements represents the precision of those measurements. When deciding whether

measurements agree with a theoretical prediction, the standard deviation of those measurements is very important: if the mean of the measurements is too far away from the prediction (with the distance measured in standard deviations), then we consider the measurements as contradicting the prediction. This makes sense since they fall outside the range of values that could reasonably be expected to occur if the prediction were correct.

Exercise 11 – 4: Variance and standard deviation

1. Bridget surveyed the price of petrol at petrol stations in Cape Town and Durban. The data, in rands per litre, are given below.

Cape Town	3,96	3,76	4,00	3,91	3,69	3,72
Durban	3,97	3,81	3,52	4,08	3,88	3,68

   a) Find the mean price in each city and then state which city has the lowest mean.
   b) Find the standard deviation of each city's prices.
   c) Which city has the more consistently priced petrol? Give reasons for your answer.

2. Compute the mean and variance of the following set of values.
   150 ; 300 ; 250 ; 270 ; 130 ; 80 ; 700 ; 500 ; 200 ; 220 ; 110 ; 320 ; 420 ; 140

3. Compute the mean and variance of the following set of values.
   −6,9 ; −17,3 ; 18,1 ; 1,5 ; 8,1 ; 9,6 ; −13,1 ; −14,0 ; 10,5 ; −14,8 ; −6,5 ; 1,4

4. The times for 8 athletes who ran a 100 m sprint on the same track are shown below. All times are in seconds.
   10,2 ; 10,8 ; 10,9 ; 10,3 ; 10,2 ; 10,4 ; 10,1 ; 10,4
   a) Calculate the mean time.
   b) Calculate the standard deviation for the data.
   c) How many of the athletes' times are more than one standard deviation away from the mean?

5. The following data set has a mean of 14,7 and a variance of 10,01.

$$18 \ ; \ 11 \ ; \ 12 \ ; \ a \ ; \ 16 \ ; \ 11 \ ; \ 19 \ ; \ 14 \ ; \ b \ ; \ 13$$

   Compute the values of a and b.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

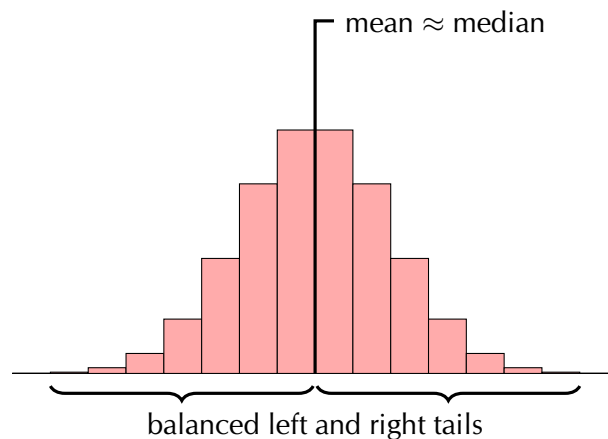
1. 23CX   2. 23CY   3. 23CZ   4. 23D2   5. 23D3

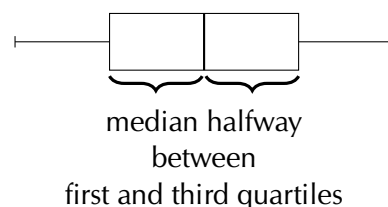
11.5 Symmetric and skewed data EMBKD

We are now going to classify data sets into 3 categories that describe the shape of the data distribution: symmetric, left skewed, right skewed. We can use this classification for any data set, but here we will look only at distributions with one peak. Most of the data distributions that you have seen so far have only one peak, so the plots in this section should look familiar.

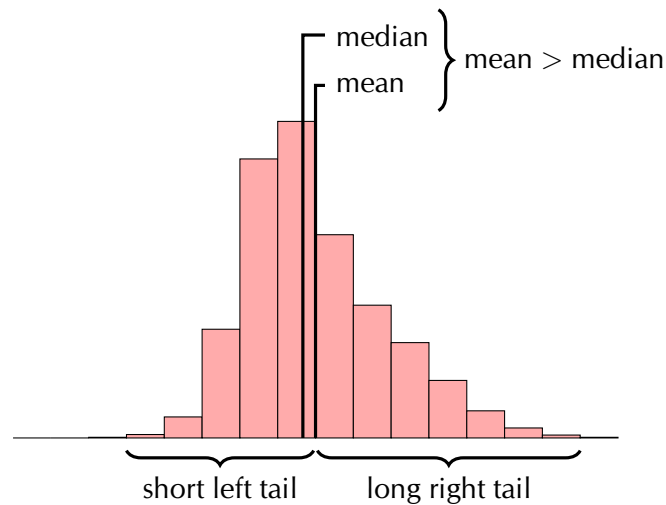
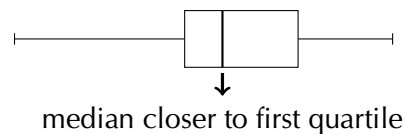
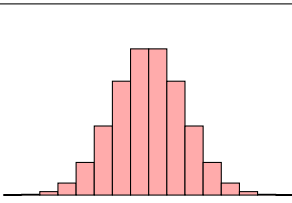
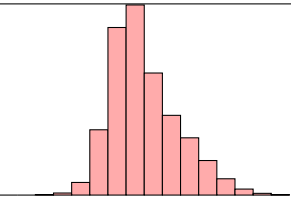
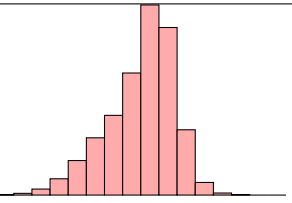
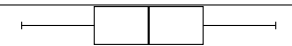
Distributions with one peak are called **unimodal distributions**. Unimodal literally means having one mode. (Remember that a mode is a maximum in the distribution.)

Symmetric distributions EMBKF

A symmetric distribution is one where the left and right hand sides of the distribution are roughly equally balanced around the mean. The histogram below shows a typical symmetric distribution.

For symmetric distributions, the mean is approximately equal to the median. The **tails** of the distribution are the parts to the left and to the right, away from the mean. The tail is the part where the counts in the histogram become smaller. For a symmetric distribution, the left and right tails are equally balanced, meaning that they have about the same length.

The figure below shows the box and whisker diagram for a typical symmetric data set.

Another property of a symmetric distribution is that its median (second quartile) lies in the middle of its first and third quartiles. Note that the whiskers of the plot (the minimum and maximum) do not have to be equally far away from the median. In the next section on outliers, you will see that the minimum and maximum values do not necessarily match the rest of the data distribution well.

Skewed

A distribution that is **skewed right** (also known as **positively skewed**) is shown below.

Now the picture is not symmetric around the mean anymore. For a right skewed distribution, the mean is typically greater than the median. Also notice that the tail of the distribution on the right hand (positive) side is longer than on the left hand side.

From the box and whisker diagram we can also see that the median is closer to the first quartile than the third quartile. The fact that the right hand side tail of the distribution is longer than the left can also be seen.

A distribution that is skewed left has exactly the opposite characteristics of one that is skewed right:

- the mean is typically less than the median;
- the tail of the distribution is longer on the left hand side than on the right hand side; and
- the median is closer to the third quartile than to the first quartile.

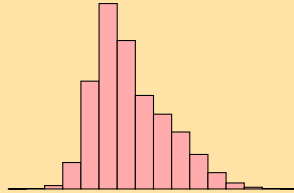
The table below summarises the different categories visually.

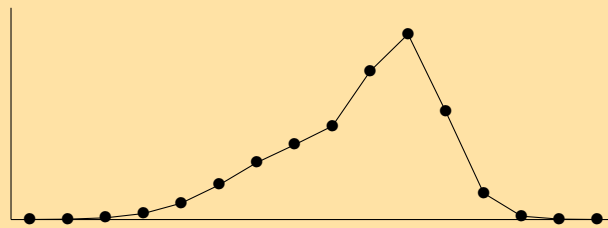
Exercise 11 – 5: Symmetric and skewed data

1. Is the following data set symmetric, skewed right or skewed left? Motivate your answer.

   27 ; 28 ; 30 ; 32 ; 34 ; 38 ; 41 ; 42 ; 43 ; 44 ; 46 ; 53 ; 56 ; 62

2. State whether each of the following data sets are symmetric, skewed right or skewed left.

   a) A data set with this histogram:

   b) A data set with this box and whisker plot:

   c) A data set with this frequency polygon:

   d) The following data set:

   11,2 ; 5 ; 9,4 ; 14,9 ; 4,4 ; 18,8 ; −0,4 ; 10,5 ; 8,3 ; 17,8

3. Two data sets have the same range and interquartile range, but one is skewed right and the other is skewed left. Sketch the box and whisker plot for each of these data sets. Then, invent data (6 points in each data set) that matches the descriptions of the two data sets.

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 23D4 2a. 23D5 2b. 23D6 2c. 23D7 2d. 23D8 3. 23D9

An outlier in a data set is a value that is far away from the rest of the values in the data set. In a box and whisker diagram, outliers are usually close to the whiskers of the diagram. This is because the centre of the diagram represents the data between the first and third quartiles, which is where 50% of the data lie, while the whiskers represent the extremes — the minimum and maximum — of the data.

Worked example 12: Identifying outliers

QUESTION

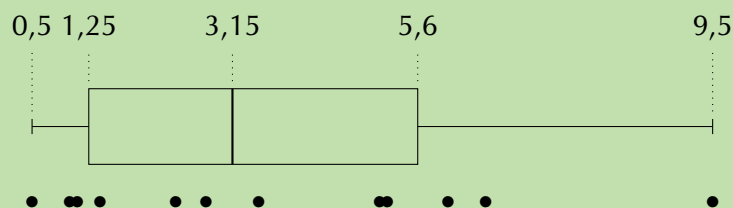
Find the outliers in the following data set by drawing a box and whisker diagram and locating the data values on the diagram.

0,5 ; 1 ; 1,1 ; 1,4 ; 2,4 ; 2,8 ; 3,5 ; 5,1 ; 5,2 ; 6 ; 6,5 ; 9,5

SOLUTION

Step 1: Determine the five number summary

The minimum of the data set is 0,5. The maximum of the data set is 9,5. Since there are 12 values in the data set, the median lies between the sixth and seventh values, making it equal to $\frac{2,8+3,5}{2} = 3,15$. The first quartile lies between the third and fourth values, making it equal to $\frac{1,1+1,4}{2} = 1,25$. The third quartile lies between the ninth and tenth values, making it equal to $\frac{5,2+6}{2} = 5,6$.

Step 2: Draw the box and whisker diagram

In the figure above, each value in the data set is shown with a black dot.

Step 3: Find the outliers

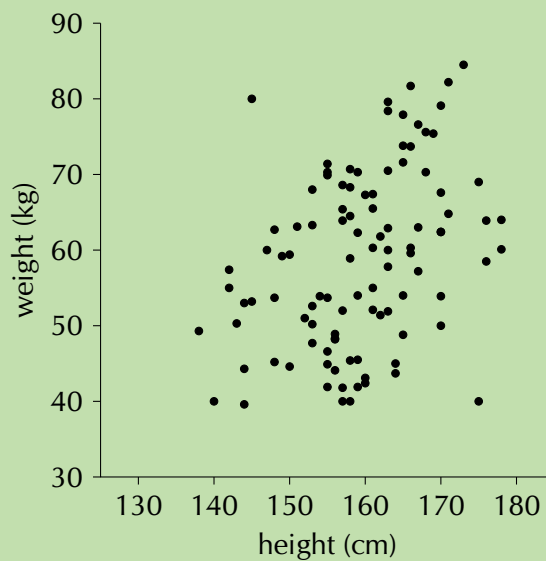
From the diagram we can see that most of the values are between 1 and 6. The only value that is very far away from this range is the maximum at 9,5. Therefore 9,5 is the only outlier in the data set.

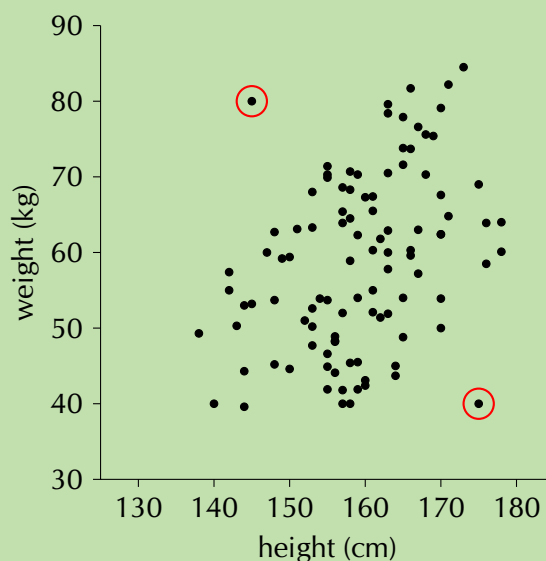
You should also be able to identify outliers in plots of two variables. A **scatter plot** is a graph that shows the relationship between two random variables. We call these data **bivariate** (literally meaning two variables) and we plot the data for two different variables on one set of axes. The following example shows what a typical scatter plot looks like. For Grade 11 you do not need to learn how to draw these 2-dimensional

scatter plots, but you should be able to identify outliers on them. As before, an outlier is a value that is far removed from the main distribution of data.

Worked example 13: Scatter plot

QUESTION

We have a data set that relates the heights and weights of a number of people. The height is the first variable and its value is plotted along the horizontal axis. The weight is the second variable and its value is plotted along the vertical axis. The data values are shown on the plot below. Identify any outliers on the scatter plot.

SOLUTION

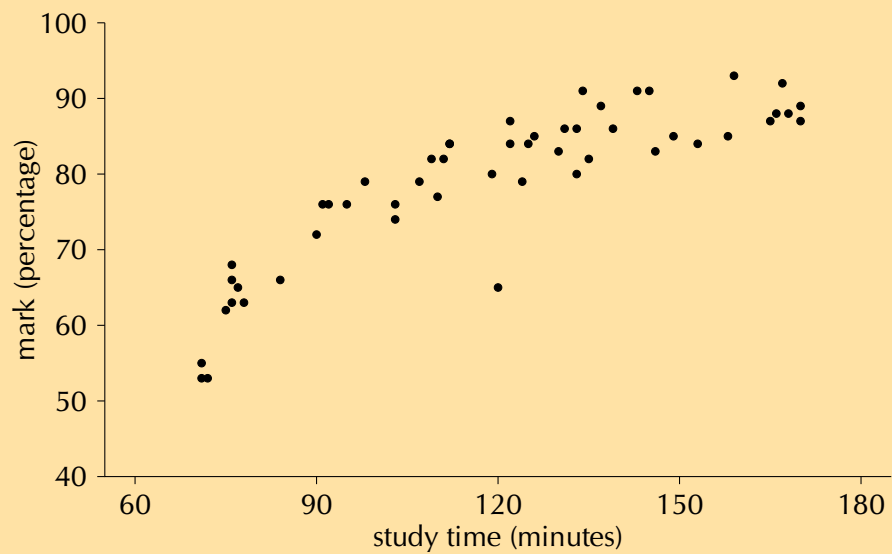
We inspect the plot visually and notice that there are two points that lie far away from the main data distribution. These two points are circled in the plot below.

Exercise 11 – 6: Outliers

1. For each of the following data sets, draw a box and whisker diagram and determine whether there are any outliers in the data.

   a) 30 ; 21,4 ; 39,4 ; 33,4 ; 21,1 ; 29,3 ; 32,8 ; 31,6 ; 36 ; 27,9 ; 27,3 ; 29,4 ; 29,1 ; 38,6 ; 33,8 ; 29,1 ; 37,1

   b) 198 ; 166 ; 175 ; 147 ; 125 ; 194 ; 119 ; 170 ; 142 ; 148

   c) 7,1 ; 9,6 ; 6,3 ; −5,9 ; 0,7 ; −0,1 ; 4,4 ; −11,7 ; 10 ; 2,3 ; −3,7 ; 5,8 ; −1,4 ; 1,7 ; −0,7

2. A class's results for a test were recorded along with the amount of time spent studying for it. The results are given below. Identify any outliers in the data.

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1a. 23DB 1b. 23DC 1c. 23DD 2. 23DF

11.7 Summary

EMBKJ

See presentation: 23DG at www.everythingmaths.co.za

- Histograms visualise how many times different events occurred. Each rectangle in a histogram represents one event and the height of the rectangle is relative to the number of times that the event occurred.
- Frequency polygons represent the same information as histograms, but using lines and points rather than rectangles. A frequency polygon connects the middle of the top edge of each rectangle in a histogram.
- Ogives (also known as cumulative histograms) show the total number of times that a value or anything less than that value appears in the data set. To draw an ogive you need to add up all the counts in a histogram from left to right.
  - The first count in an ogive is always zero.
  - The last count in an ogive is always the sum of all the counts in the data set.
- The variance and standard deviation are measures of dispersion.
  - The standard deviation is the square root of the variance.
  - Variance: $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$
  - Standard deviation: $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2}$
  - The standard deviation is measured in the same units as the mean and the data, but the variance is not. The variance is measured in the square of the data units.
- In a symmetric distribution
  - the mean is approximately equal to the median; and
  - the tails of the distribution are balanced.
- In a right (positively) skewed distribution
  - the mean is greater than the median;
  - the tail on the right hand side is longer than the tail on the left hand side; and
  - the median is closer to the first quartile than the third quartile.
- In a left (negatively) skewed distribution
  - the mean is less than the median;
  - the tail on the left hand side is longer than the tail on the right hand side; and
  - the median is closer to the third quartile than the first quartile.
- An outlier is a value that is far away from the rest of the data.

Exercise 11 – 7: End of chapter exercises

1. Draw a histogram, frequency polygon and ogive of the following data set. To count the data, use intervals with a width of 1, starting from 0.

   0,4 ; 3,1 ; 1,1 ; 2,8 ; 1,5 ; 1,3 ; 2,8 ; 3,1 ; 1,8 ; 1,3 ;

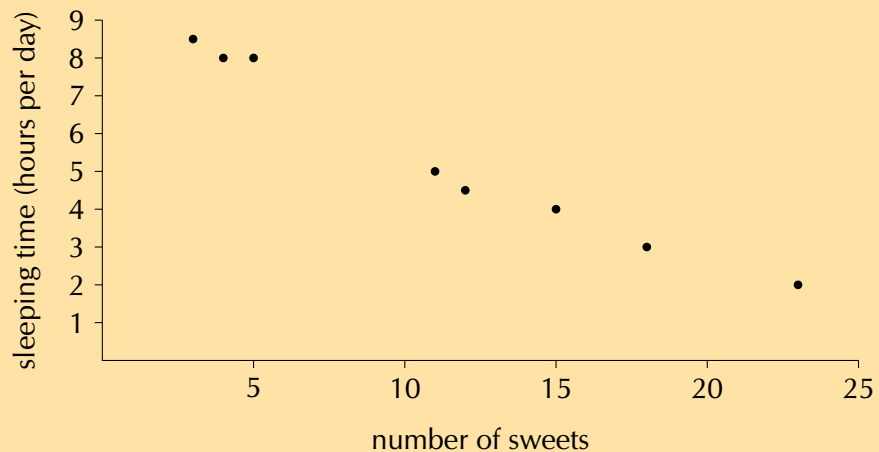
   2,6 ; 3,7 ; 3,3 ; 5,7 ; 3,7 ; 7,4 ; 4,6 ; 2,4 ; 3,5 ; 5,3

2. Draw a box and whisker diagram of the following data set and explain whether it is symmetric, skewed right or skewed left.

   −4,1 ; −1,1 ; −1 ; −1,2 ; −1,5 ; −3,2 ; −4 ; −1,9 ; −4 ;

   −0,8 ; −3,3 ; −4,5 ; −2,5 ; −4,4 ; −4,6 ; −4,4 ; −3,3

3. Eight children's sweet consumption and sleeping habits were recorded. The data are given in the following table and scatter plot.

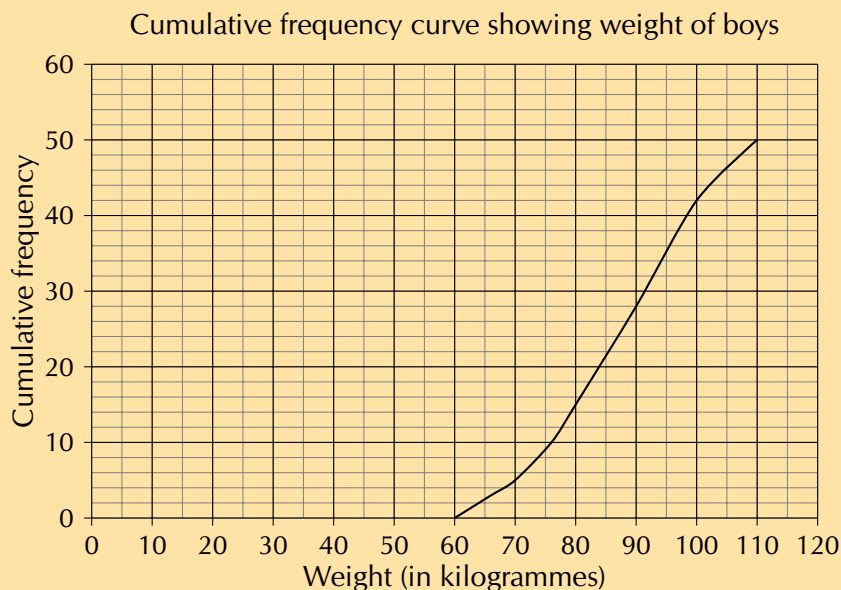
Number of sweets per week	15	12	5	3	18	23	11	4
Average sleeping time (hours per day)	4	4,5	8	8,5	3	2	5	8

   a) What is the mean and standard deviation of the number of sweets eaten per day?
   b) What is the mean and standard deviation of the number of hours slept per day?
   c) Make a list of all the outliers in the data set.

4. The monthly incomes of eight teachers are as follows:

   R 10 050; R 14 300; R 9800; R 15 000; R 12 140; R 13 800; R 11 990; R 12 900.

   a) What is the mean and standard deviation of their incomes?
   b) How many of the salaries are less than one standard deviation away from the mean?
   c) If each teacher gets a bonus of R 500 added to their pay what is the new mean and standard deviation?
   d) If each teacher gets a bonus of 10% on their salary what is the new mean and standard deviation?
   e) Determine for both of the above, how many salaries are less than one standard deviation away from the mean.

f) Using the above information work out which bonus is more beneficial financially for the teachers.

5. The weights of a random sample of boys in Grade 11 were recorded. The cumulative frequency graph (ogive) below represents the recorded weights.

a) How many of the boys weighed between 90 and 100 kilogrammes?

b) Estimate the median weight of the boys.

c) If there were 250 boys in Grade 11, estimate how many of them would weigh less than 80 kilogrammes?

6. Three sets of 12 learners each had their test scores recorded. The test was out of 50. Use the given data to answer the following questions.

Set A	Set B	Set C
25	32	43
47	34	47
15	35	16
17	32	43
16	25	38
26	16	44
24	38	42
27	47	50
22	43	50
24	29	44
12	18	43
31	25	42

a) For each of the sets calculate the mean and the five number summary.

b) Make box and whisker plots of the three data sets on the same set of axes.

c) State, with reasons, whether each of the three data sets are symmetric or skewed (either right or left).

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1. 23DH 2. 23DJ 3. 23DK 4. 23DM 5. 23DN 6. 23DP

CHAPTER 12

Linear programming

12.1 Introduction EMBKK

In everyday life people are interested in knowing the most efficient way of carrying out a task or achieving a goal. For example, a farmer wants to know how many hectares to plant during a season in order to maximise the yield (produce), a stock broker wants to know how much to invest in stocks in order to maximise profit, an entrepreneur wants to know how many people to employ to minimise expenditure. These are optimisation problems; we want to to determine either the maximum or the minimum in a specific situation.

To describe this mathematically, we assign variables to represent the different factors that influence the situation. Optimisation means finding the combination of variables that gives the best result.

See video: 23DQ at www.everythingmaths.co.za

Worked example 1: Mountees and Roadees

QUESTION

Investigate the following situation and use your knowledge of mathematics to solve the problem:

Mr. Hunter manufactures bicycles. He produces two different models of bicycles; strong mountain bikes called Mountees and fast road bikes called Roadees. He cannot produce more than 5 Mountees on a day and he can manufacture a maximum of 3 Roadees a day. He needs 1 technician to assemble a Mountee and 2 technicians to assemble a Roadee. The company has 8 technicians in the assembly department. The profit on a Mountee is R 800 and R 2400 on a Roadee. The demand is such that he can sell all the bikes he manufactures.

Determine the number of each model of bicycle that must be manufactured in order to make the maximum profit.

SOLUTION

Step 1: Assign variables

In this situation there are two variables that we need to consider: let the number of Mountees produced be M and let the number of Roadees produced be R.

Step 2: Organise the information given

Write down a summary of the information given in the problem so that we consider

all the different components in the situation.

$$\begin{aligned}
\text{maximum number for } M &= 5 \\
\text{maximum number for } R &= 3 \\
\text{number of technicians needed for } M &= 1 \\
\text{number of technicians needed for } R &= 2 \\
\text{total number of technicians} &= 8 \\
\text{profit per } M &= 800 \\
\text{profit per } R &= 2400
\end{aligned}$$

Step 3: Draw up a table

Use the summary to draw up a table of all the possible combinations of the number of Mountees and Roadees that can be manufactured per day:

M	R			
	0	1	2	3
0	$(0;0)$	$(0;1)$	$(0;2)$	$(0;3)$
1	$(1;0)$	$(1;1)$	$(1;2)$	$(1;3)$
2	$(2;0)$	$(2;1)$	$(2;2)$	$(2;3)$
3	$(3;0)$	$(3;1)$	$(3;2)$	$(3;3)$
4	$(4;0)$	$(4;1)$	$(4;2)$	$(4;3)$
5	$(5;0)$	$(5;1)$	$(5;2)$	$(5;3)$

Note that there are 24 possible combinations.

Step 4: Consider the limitation of the number of technicians

It takes 1 technician to assemble a Mountee and 2 technicians to assemble a Roadee. There are a total of 8 technicians in the assembly department, therefore we can write that $1(M) + 2(R) \leq 8$.

With this limitation, we are able to eliminate some of the combinations in the table where $M + 2R > 8$:

M	R			
	0	1	2	3
0	$(0;0)$	$(0;1)$	$(0;2)$	$(0;3)$
1	$(1;0)$	$(1;1)$	$(1;2)$	$(1;3)$
2	$(2;0)$	$(2;1)$	$(2;2)$	$(2;3)$
3	$(3;0)$	$(3;1)$	$(3;2)$	~~$(3;3)$~~
4	$(4;0)$	$(4;1)$	$(4;2)$	~~$(4;3)$~~
5	$(5;0)$	$(5;1)$	~~$(5;2)$~~	~~$(5;3)$~~

These combinations have been excluded as possible answers. For example, $(5;3)$ gives $5 + 2(3) = 11$ technicians.

Step 5: Consider the profit on the bicycles

We can express the profit (P) per day as: $P = 800(M) + 2400(R)$. Notice that a higher profit is made on a Roadee.

By substituting the different combinations for M and R, we can find the values that give the maximum profit:

$$\begin{aligned} \text{For } (5;0) \quad P &= 800(5) + 2400(0) \\ &= \text{R } 4000 \\ \text{For } (3;1) \quad P &= 800(3) + 2400(1) \\ &= \text{R } 4800 \end{aligned}$$

M	R			
	0	1	2	3
0	$(0;0) \Rightarrow$ R 0	$(0;1) \Rightarrow$ R 2400	$(0;2) \Rightarrow$ R 4800	$(0;3) \Rightarrow$ R 7200
1	$(1;0) \Rightarrow$ R 800	$(1;1) \Rightarrow$ R 3200	$(1;2) \Rightarrow$ R 5600	$(1;3) \Rightarrow$ R 8000
2	$(2;0) \Rightarrow$ R 1600	$(2;1) \Rightarrow$ R 4000	$(2;2) \Rightarrow$ R 6400	$(2;3) \Rightarrow$ R 8800
3	$(3;0) \Rightarrow$ R 2400	$(3;1) \Rightarrow$ R 4800	$(3;2) \Rightarrow$ R 7200	~~$(3;3)$~~
4	$(4;0) \Rightarrow$ R 3200	$(4;1) \Rightarrow$ R 5600	$(4;2) \Rightarrow$ R 8000	~~$(4;3)$~~
5	$(5;0) \Rightarrow$ R 4000	$(5;1) \Rightarrow$ R 6400	~~$(5;2)$~~	~~$(5;3)$~~

Step 6: Write the final answer

Therefore the maximum profit of R 8800 is obtained if 2 Mountees and 3 Roadees are manufactured per day.

Exercise 12 – 1: Optimisation

1. Furniture store opening special:

   As part of their opening special, a furniture store has promised to give away at least 40 prizes with a total value of at least R 4000. They intend to give away kettles and toasters. They decide there will be at least 10 units of each prize. A kettle costs the company R 120 and a toaster costs R 100.

   Determine how many of each prize will represent the cheapest option for the company. Calculate how much this combination of kettles and toasters will cost.

   Use a suitable strategy to organise the information and solve the problem.

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1. 23DR

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Optimisation using graphs

A more efficient way to solve optimisation problems is using graphs.

We write the limitations in the situation, called constraints, as inequalities. Some constraints can be modelled by an equation, which needs to be maximised or minimized. We sketch the inequalities and indicate the region above or below the line that is to be considered in determining the solution. This method of solving optimisation problems is called linear programming.

See video: 23DS at www.everythingmaths.co.za

Worked example 2: Optimisation using graphs

QUESTION

Consider again the example of Mr. Hunter who manufactures Mountees and Roadees:

Mr. Hunter manufactures bicycles. He produces two different models of bicycles; strong mountain bikes called Mountees and fast road bikes called Roadees. He cannot produce more than 5 Mountees on a day and he can manufacture a maximum of 3 Roadees a day. He needs 1 technician to assemble a Mountee and 2 technicians to assemble a Roadee. The company has 8 technicians in the assembly department. The profit on a Mountee is R 800 and R 2400 on a Roadee. The demand is such that he can sell all the bikes he manufactures.

Determine the number of each model of bicycle that must be manufactured in order to make a maximum profit.

SOLUTION

Step 1: Assign variables

In this situation there are two variables that we need to consider: let the number of Mountees produced be M and let the number of Roadees produced be R.

Notice that the values of M and R are limited to positive integers; Mr. Hunter cannot sell negative numbers of bikes nor can he sell a fraction of a bike.

Step 2: Organise the information

We can write these constraints as inequalities:

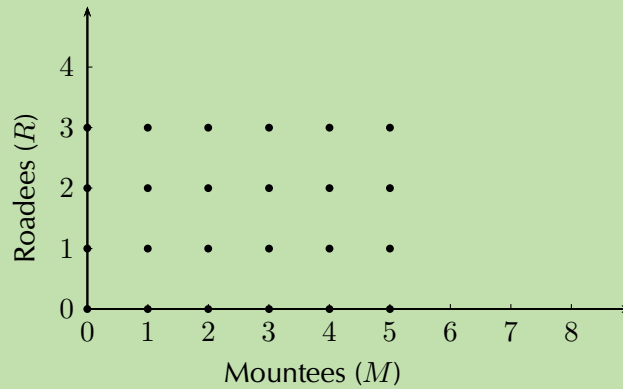
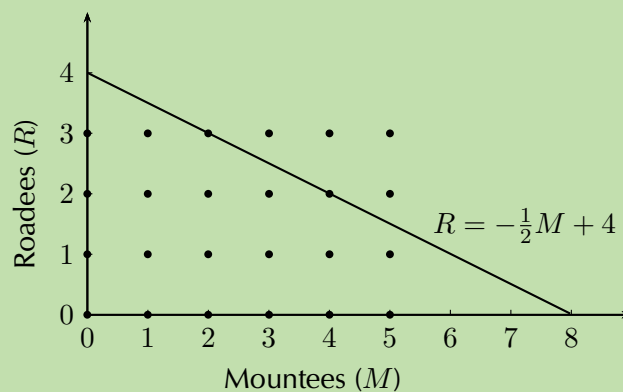
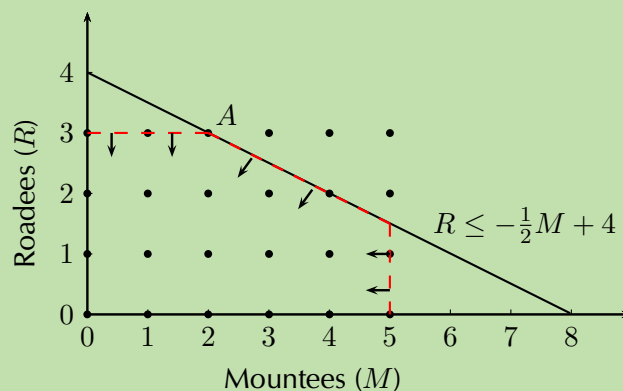
$$\text{number of Mountees: } 0 \leq M \leq 5$$

$$\text{number of Roadees: } 0 \leq R \leq 3$$

$$\text{total number of technicians: } M + 2R \leq 8$$

We also know that $P = 800M + 2400R$. This is called the objective function, sometimes also referred to as the search line, because the objective (goal) is to determine the maximum value of P.

Step 3: Solve using graphs

We represent the number of Mountees manufactured daily on the horizontal axis and the number of Roadees manufactured daily on the vertical axis. Since M and R are positive integers, we only use the first quadrant of the Cartesian plane. Note that the graph only includes the integer values of M between 0 and 5 and R between 0 and 3.

For the number of technicians in the assembly department $M + 2R \leq 8$. If we make R (represented on the y-axis) the subject of the inequality we get $R \leq -\frac{1}{2}M + 4$.

The arrows indicate the region in which the solution will lie, where $R \leq -\frac{1}{2}M + 4$. This area is called the feasible region.

We substitute the possible combinations into the profit equation $P = 800M + 2400R$, and find the combination that gives the maximum profit.

$$\begin{aligned} \text{At } A(2;3): \quad P &= 800(2) + 2400(3) \\ &= \text{R } 8800 \end{aligned}$$

Step 4: Write the final answer

Therefore the maximum profit is obtained if 2 Mountees and 3 Roadees are manufactured per day.

See video: 23DT at www.everythingmaths.co.za

Worked example 3: Optimisation using graphs

QUESTION

Solve the "furniture store opening special" problem using graphs:

As part of their opening special, a furniture store has promised to give away at least 40 prizes. They intend to give away kettles and toasters. They decide there will be at least 10 units of each prize. A kettle costs the company R 120 and a toaster costs R 100.

Determine how many of each prize will represent the cheapest option for the company. Calculate how much this combination of kettles and toasters will cost.

SOLUTION

Step 1: Assign variables

In this situation there are two variables that we need to consider: let the number of kettles be k and the number of toasters be t, with $k, t \in \mathbb{Z}$.

Step 2: Organise the information

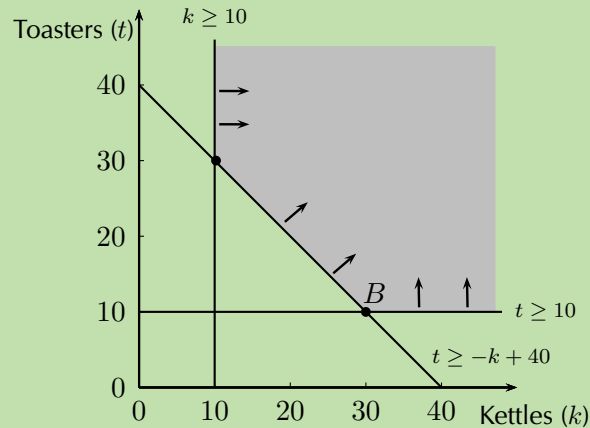
We can write the given information as inequalities:

$$\begin{aligned} \text{number of kettles: } k &\geq 10 \\ \text{number of toasters: } t &\geq 10 \\ \text{total number of prizes: } k + t &\geq 40 \end{aligned}$$

We make t the subject of the inequality:

$$t \geq -k + 40$$

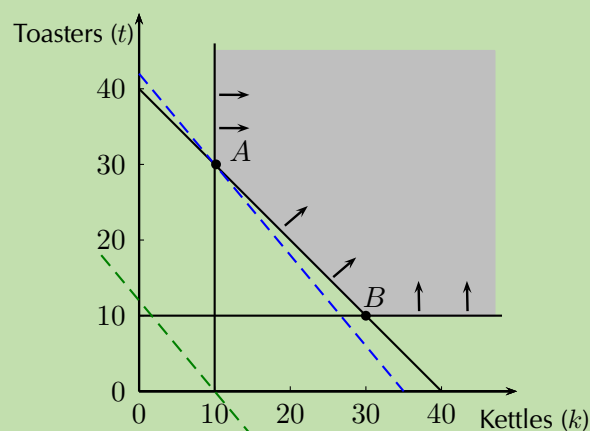
Step 3: Solve using graphs

Represent the constraints on a set of axes:

We shade the feasible region as shown in the diagram. Remember that in this situation only the points with integer coordinates inside or on the border of the feasible region are possible solutions. The combination giving the minimum cost will lie towards or on the lower border of the feasible region, which gives us many points to consider. To find the optimum value of C, we use the graph of the objective function

$$C = 120k + 100t$$

To draw the line, we make t the subject of the formula

$$t = -\frac{6}{5}k + \frac{C}{100}$$

We see that the gradient of the objective function is $-\frac{6}{5}$, but we do not know the exact value of the t-intercept ($\frac{C}{100}$). To find the minimum value of C, we need to determine the position of the objective function where it first touches the feasible region and also gives the lowest t-intercept.

We indicate the gradient of the objective function on the graph (the green search line). Keeping the gradient the same, we "slide" the objective function towards the lower border of the feasible region and find that it touches the feasible region at point

$A(10;30)$. This optimum position of the objective function is indicated on the graph by the dotted line passing through point A.

We substitute the coordinates of A into the cost equation $C = 120k + 100t$:

$$\begin{aligned} \text{At } A(10;30): \quad C &= 120(10) + 100(30) \\ &= \text{R } 4200 \end{aligned}$$

The minimum cost can also be determined graphically by reading off the coordinates of the t-intercept of the objective function in the optimum position:

$$\begin{aligned} t_{\text{int}} &= 42 \\ \therefore \frac{C}{100} &= 42 \\ \therefore C &= \text{R } 4200 \end{aligned}$$

Step 4: Write the final answer

Therefore the minimum cost to the company is R 4200 with 10 kettles and 30 toasters.

Exercise 12 – 2: Optimisation

1. You are given a test consisting of two sections. The first section is on algebra and the second section is on geometry. You are not allowed to answer more than 10 questions from any section, but you have to answer at least 4 algebra questions. The time allowed is not more than 30 minutes. An algebra problem will take 2 minutes and a geometry problem will take 3 minutes to solve.

   Let x be the number of algebra questions and y be the number of geometry questions.

   a) Formulate the equations and inequalities that satisfy the above constraints.
   b) The algebra questions carry 5 marks each and the geometry questions carry 10 marks each. If T is the total marks, write down an expression for T.

2. A local clinic wants to produce a guide to healthy living. The clinic intends to produce the guide in two formats: a short video and a printed book. The clinic needs to decide how many of each format to produce for sale. Estimates show that no more than 10 000 copies of both items together will be sold. At least 4000 copies of the video and at least 2000 copies of the book could be sold, although sales of the book are not expected to exceed 4000 copies. Let x be the number of videos sold, and y the number of printed books sold.

   a) Write down the constraint inequalities that can be deduced from the given information.
   b) Represent these inequalities graphically and indicate the feasible region clearly.

c) The clinic is seeking to maximise the income, I, earned from the sales of the two products. Each video will sell for R 50 and each book for R 30. Write down the objective function for the income.

d) What maximum income will be generated by the two guides?

3. A certain motorcycle manufacturer produces two basic models, the Super X and the Super Y. These motorcycles are sold to dealers at a profit of R 20 000 per Super X and R 10 000 per Super Y. A Super X requires 150 hours for assembly, 50 hours for painting and finishing and 10 hours for checking and testing. The Super Y requires 60 hours for assembly, 40 hours for painting and finishing and 20 hours for checking and testing. The total number of hours available per month is: 30 000 in the assembly department, 13 000 in the painting and finishing department and 5000 in the checking and testing department.

   The above information is summarised by the following table:

Department	Hours for Super X	Hours for Super Y	Hours available per month
Assembly	150	60	30 000
Painting and finishing	50	40	13 000
Checking and testing	10	20	5000

   Let x be the number of Super X and y be the number of Super Y models manufactured per month.

   a) Write down the set of constraint inequalities.
   b) Use graph paper to represent the set of constraint inequalities.
   c) Shade the feasible region on the graph paper.
   d) Write down the profit generated in terms of x and y.
   e) How many motorcycles of each model must be produced in order to maximise the monthly profit?
   f) What is the maximum monthly profit?

4. A group of students plan to sell x hamburgers and y chicken burgers at a rugby match. They have meat for at most 300 hamburgers and at most 400 chicken burgers. Each burger of both types is sold in a packet. There are 500 packets available. The demand is likely to be such that the number of chicken burgers sold is at least half the number of hamburgers sold.

   a) Write the constraint inequalities and draw a graph of the feasible region.
   b) A profit of R 3 is made on each hamburger sold and R 2 on each chicken burger sold. Write the equation which represents the total profit P in terms of x and y.
   c) The objective is to maximise profit. How many of each type of burger should be sold?

5. Fashion-Cards is a small company that makes two types of cards, type X and type Y. With the available labour and material, the company can make at most 150 cards of type X and at most 120 cards of type Y per week. Altogether they cannot make more than 200 cards per week.

There is an order for at least 40 type X cards and 10 type Y cards per week. Fashion-Cards makes a profit of R 5 for each type X card sold and R 10 for each type Y card.

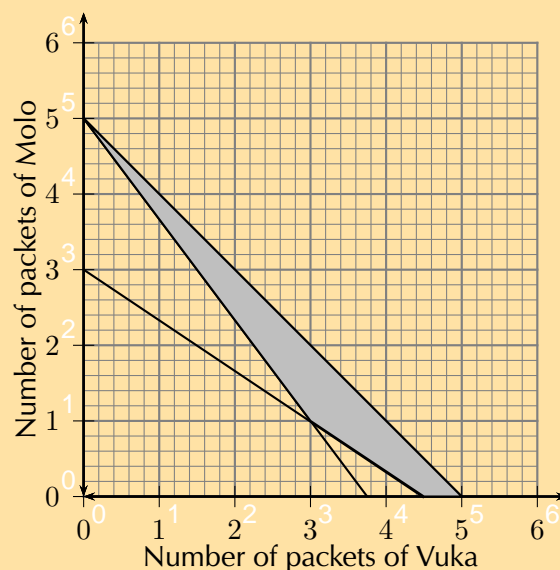
Let the number of type X cards manufactured per week be x and the number of type Y cards manufactured per week be y.

a) One of the constraint inequalities which represents the restrictions above is $0 \leq x \leq 150$. Write the other constraint inequalities.

b) Represent the constraints graphically and shade the feasible region.

c) Write the equation that represents the profit P (the objective function), in terms of x and y.

d) Calculate the maximum weekly profit.

6. To meet the requirements of a specialised diet a meal is prepared by mixing two types of cereal, Vuka and Molo. The mixture must contain x packets of Vuka cereal and y packets of Molo cereal. The meal requires at least 15 g of protein and at least 72 g of carbohydrates. Each packet of Vuka cereal contains 4 g of protein and 16 g of carbohydrates. Each packet of Molo cereal contains 3 g of protein and 24 g of carbohydrates. There are at most 5 packets of cereal available. The feasible region is shaded on the attached graph paper.

a) Write down the constraint inequalities.

b) If Vuka cereal costs R 6 per packet and Molo cereal also costs R 6 per packet, use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a minimum.

c) Use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a maximum (give all possibilities).

Think you got it? Get this answer and more practice on our Intelligent Practice Service

1. 23DV 2. 23DW 3. 23DX 4. 23DY 5. 23DZ 6. 23F2

Solutions to exercises

1 Exponents and surds

Exercise 1 – 1: The number system

1. $\mathbb{R};\mathbb{Q}'$
2. $\mathbb{R};\mathbb{Q}$
3. $\mathbb{R};\mathbb{Q}$
4. $\mathbb{R};\mathbb{Q}$
5. $\mathbb{R};\mathbb{Q};\mathbb{Z};\mathbb{N}_0$
6. $\mathbb{R}'\mathbb{Q}'$
7. $\mathbb{R};\mathbb{Q}$
8. $\mathbb{R};\mathbb{Q}'$
9. $\mathbb{R}'$
10. $\mathbb{R};\mathbb{Q}'$
11. $\mathbb{R};\mathbb{Q}$
12. $\mathbb{R};\mathbb{Q};\mathbb{Z}$
13. $\mathbb{R};\mathbb{Q}$
14. $\mathbb{R};\mathbb{Q}'$
15. $\mathbb{R};\mathbb{Q}$
16. $\mathbb{R};\mathbb{Q};\mathbb{Z}$

Exercise 1 – 2: Laws of exponents

1. 4^{3a+3}
2. 72
3. $9p^{10}$
4. k^{2x-2}
5. $5^{2z-2}+5^{z}$
6. 1
7. x^{10}
8. $\frac{b^2}{a^2}$
9. $\frac{1}{m+n}$
10. $2p^{ts}$
11. $\frac{1}{a}$
12. k
13. 2^{a+1}
14. h^4
15. $\frac{a^4b^6}{c^6d^2}$
16. 4
17. $\frac{m^2n^2}{2}$
18. 400
19. $\frac{1}{y^7}$
20. 8
21. 2^{6a+2}
22. $2pt$
23. $81q^{2s}y^{8a+2}$

Exercise 1 – 3: Rational exponents and surds

1. a) 7
   b) $\frac{1}{6}$
   c) $\frac{1}{\sqrt[3]{36}}$
   d) $-\frac{4}{3}$
   e) $8x^3$
2. a) $s^{\frac{1}{6}}$
   b) $16m^4$
   c) $\frac{3}{2}m^2$
   d) 8
3. $x^{\frac{31}{16}}$

Exercise 1 – 4: Simplification of surds

1. a) 4
   b) ab^4c^2
   c) 2
   d) xy^4
2. a) $\frac{ab}{b-a}$
   b) $-\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)$

Exercise 1 – 5: Rationalising the denominator

1. $2\sqrt{5}$
2. $\frac{\sqrt{6}}{2}$
3. $\sqrt{6}$
4. $\dfrac{3\sqrt{5}+3}{4}$
5. $\dfrac{x\sqrt{y}}{y}$
6. $\dfrac{\sqrt{6}+\sqrt{14}}{2}$
7. $\dfrac{3p-4\sqrt{p}}{p}$
8. $\sqrt{t}-2$
9. $\frac{1-\sqrt{m}}{1-m}$
10. $\sqrt{ab}$

Exercise 1 – 6: Solving surd equations

1. $x=4$
2. $p=3$
3. $y=1$
4. $t=3$
5. $z=9$ or $z=\frac{1}{4}$
6. $x=8$ or $x=-27$
7. $n=-\frac{1}{4}$
8. $d=3$ or $d=-5$
9. $y=1$ or $y=81$
10. $f=5$

Exercise 1 – 7: Applications of exponentials

1. 9,7%
2. 4 254 691
3. 7
4. 26 893

Exercise 1 – 8: End of chapter exercises

1. a) $\frac{1}{4}$
   b) $4\frac{1}{4}$
2. a) x^4
   b) s
   c) $m^{\frac{25}{3}}$
   d) $m^{\frac{8}{3}}$
   e) $-m^{\frac{8}{3}}$
   f) $81y^{\frac{16}{3}}$
3. a) $\frac{3b^{\frac{45}{2}}}{(a^{12}c^{\frac{5}{2}}}$
   b) $3a^3b^2$
   c) $a^{24}b^{12}$
   d) $x^{\frac{7}{2}}$
   e) $x^{\frac{4}{3}}b^{\frac{5}{3}}$
4. $\frac{1}{x^{\frac{1}{16}}}$
5. $x-2$
6. $\frac{10\sqrt{x}+10}{x-1}$
7. $\frac{3\sqrt{x}+2x\sqrt{x}}{2x}$
8. a) $6\sqrt{2}$
   b) $7\sqrt{5}$
   c) 2
   d) $\frac{1}{4\sqrt{2}}$
   e) 2
   f) $\frac{16\sqrt{15}}{5}$
9. a) $6+4\sqrt{2}$
   b) $6+5\sqrt{2}$
   c) $4+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$
10. a) 55
    b) 1
11. $15\sqrt{2}x^3$
12. a) $1+\frac{2\sqrt{5}}{5}$
    b) $\frac{2y+y\sqrt{y}-4\sqrt{y}-8}{y-4}$
    c) $2\sqrt{x}+2\sqrt{10}$
13. $\frac{3}{2}$
15. 3
16. $-\sqrt{288}$
17. a) 4
    b) $-\frac{1}{3}$
    c) 3
    d) No solution
    e) $x=\frac{1}{8}$ or $x=-8$
18. b) $x=1$

2 Equations and inequalities

Exercise 2 – 1: Solution by factorisation

1. $t=0$ or $t=-2$
2. $y=-1$
3. $s=\pm 5$
4. $y=3$ or $y=2$
5. $y=4$ or $y=-9$
6. $p=-2$
7. $y=-3$ or $y=-8$
8. $y=6$ or $y=7$
9. $x=-7$ or $x=-2$
10. $y=4k$ or $y=k$
11. $y=9$ or $y=-9$
12. $y=\pm\sqrt{5}$
13. $h=\pm 6$
14. $y=\pm\sqrt{14}$
15. $p=-2$
16. $y=\pm 6\sqrt{2}$
17. $f=\frac{5}{2}$ or $f=-3$
18. $x=\frac{1}{4}$
19. $y=\frac{1}{7}$
20. $x\in\mathbb{R}, x\neq\pm 3$
21. $y=-13$ or $y=-1$
22. $t=\frac{3}{2}$ or $t=-2$
23. $m=-6$
24. $t=0$ or $t=3$

Exercise 2 – 2: Solution by completing the square

1. a) $x=-5-3\sqrt{3}$ or $x=-5+3\sqrt{3}$
   b) $x=-1$ or $x=-3$
   c) $p=-4\pm\sqrt{21}$
   d) $x=-3\pm\sqrt{7}$
   e) No real solution
   f) $t=-8\pm 3\sqrt{6}$
   g) $x=-1\pm\sqrt{\frac{5}{3}}$
   h) $z=-4\pm\sqrt{22}$
   i) $z=\frac{11}{2}$ or $z=0$
   j) $z=5$ or $z=-1$
2. $k=-3\pm\sqrt{9-a}$
3. $y=\frac{-q\pm\sqrt{q^2-4pr}}{2p}$

Exercise 2 – 3: Solution by the quadratic formula

1. $t = 1$ or $t = \frac{-4}{3}$
2. $x = \frac{5+\sqrt{37}}{2}$ or $t = \frac{5-\sqrt{37}}{2}$
3. No real solution
4. $p = \frac{1}{2}$ or $p = -1$
5. No real solution
6. $t = \frac{-3+\sqrt{69}}{10}$ or $t = \frac{-3-\sqrt{69}}{10}$
7. $t = 2 \pm \sqrt{2}$
8. $k = \frac{7+\sqrt{373}}{18}$ or $k = \frac{7-\sqrt{373}}{18}$
9. $f = \frac{1}{2}$ or $f = -2$
10. No real solution

Exercise 2 – 4:

1. $x = -1$, $x = -4$, $x = -2$ and $x = -3$
2. $x = 1$, $x = 4$ and $x = -2$
3. $x = -7$, $x = 4$, $x = -1$ and $x = -2$
4. $x = -4$, $x = 3$, $x = -3$ and $x = 2$
5. $x = \frac{8\pm\sqrt{40}}{4}$
6. $x = -5$, $x = 3$, $x = -1 + \sqrt{10}$ and $x = -1 - \sqrt{10}$

Exercise 2 – 5: Finding the equation

1. $x^2 - x - 6 = 0$
2. $x^2 - 16 = 0$
3. $2x^2 - 5x - 3 = 0$
4. $k = 3$ and $x = \frac{3}{4}$
5. $p = 5$ and $x = -1$

Exercise 2 – 6: Mixed exercises

1. $y = \frac{1}{8}$ or $y = -\frac{8}{3}$
2. $x = \frac{3}{2}$ or $x = -\frac{7}{2}$
3. $t = \frac{2}{3}$ or $t = 2$
4. $y = 1$ or $y = -1$
5. $m = 1$ or $m = 4$
6. $y = \pm\frac{5}{7}$
7. $w = \frac{3}{2}$ or $w = 4$
8. $y = \frac{6}{5}$ or $y = \frac{1}{4}$
9. $n = \frac{8}{3}$ or $n = -\frac{9}{8}$
10. $y = -\frac{8}{3}$ or $y = \frac{3}{2}$
11. $x = -\frac{1}{2}$ or $x = 3$
12. $y = -\frac{5}{2}$ or $y = -\frac{5}{9}$
13. $y = \frac{4}{5}$ or $y = \frac{1}{5}$
14. $g = -\frac{1}{4}$ or $g = 1$
15. $y = 2$ or $y = -\frac{5}{9}$
16. $p = \frac{3}{7}$ or $p = -\frac{1}{5}$
17. $y = -\frac{2}{9}$ or $y = -1$
18. $y = \frac{9}{2}$ or $y = \frac{9}{7}$

Exercise 2 – 7: From past papers

1. a) Real, unequal and rational
   b) Real and equal
   c) Real, unequal and irrational
   d) Real, unequal and rational
   e) Real, unequal and irrational
   f) Non-real
   g) Real, unequal and rational
   h) Real, unequal and irrational
   i) Non-real
   j) Real and equal
2. b) real and unequal
   c) $k = -6 \pm 2\sqrt{6}$
4. a) $k = 6$
   b) $k = \frac{1}{3}$
5. a) $k = 4$ or $k = 1$
   b) $k = 0$ or $k = 5$
6. a) all real values of a, b and p
   b) $a = b$ and $p = 0$

Exercise 2 – 8: Solving quadratic inequalities

1. a) $-3 < x < 4$
   b) $x < -\frac{4}{3}$ or when $x > 1$
   c) no real solutions
   d) $-1 < t < 3$
   e) All real values of s.
   f) All real values of x.
   g) $x \leq -\frac{1}{4}$ or $x \geq 0$
   i) $x < 3$ or $x > 6$ with $x \neq 3$
   j) $-2 \leq x \leq 2$ and $x > 7$ with $x \neq 7$
   k) $x > 0$ with $x \neq 0$
2. a) $x < -3$ or $x > 3$
   b) $-\sqrt{5} \leq x \leq \sqrt{5}$
   c) no solution
   d) All real values of x

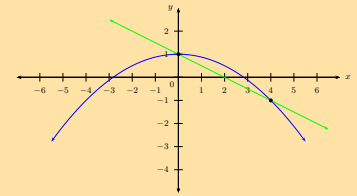
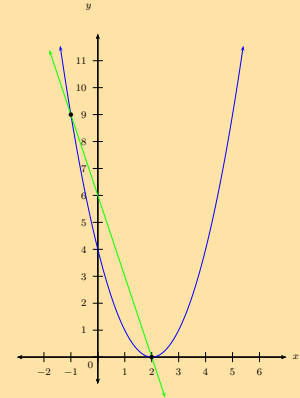
Exercise 2 – 9: Solving simultaneous equations

1. a) $(0; 5)$ and $(2; 3)$
   b) $x = 3 \pm \sqrt{2}$ and $y = 2 \pm \sqrt{2}$
   c) $(-1; 0)$ and $(\frac{1}{4}; \frac{5}{8})$
   d) $b = \frac{2 \pm \sqrt{88}}{6}$ and $a = \frac{11 \pm \sqrt{88}}{6}$
   e) $(-3; -20)$ and $(2; 0)$
   f) $x = \frac{6 \pm \sqrt{264}}{2}$ and $y = \frac{70 \pm \sqrt{264}}{2}$
2. a) $(-3; 8)$ and $(2; 3)$
   b) $(-4; 14)$ and $(3; 7)$
   c) $(3; 4)$ and $(4; 3)$

Exercise 2 – 10:

1. $b = 2$ m, $l = 4$ m
2. 187
3. $t = 10{,}5$ s
4. $t = 5d$; 105 minutes; 1,4 km
5. 24 A; 70 W; 12 A

Exercise 2 – 11: End of chapter exercises

1. $x = 1{,}62$ or $x = -0{,}62$
2. $x = \pm 4$ or $x = -1$
3. $y = 0$ or $y = \pm 1$
4. $x = \pm 2$
5. a) $x = 7$ or $x = 2$
   b) $x = 2{,}3$ or $x = -1{,}3$
   c) $x = 1{,}65$ or $x = -3{,}65$
   d) $x = 0$ or $x = -3$
6. $x = \frac{\sqrt{16+p^2}-2}{2}$
7. $a = 3$; $b = 10$ and $c = -8$
8. $p = \pm 16$
9. $x^2 + 2x - 15$
10. Undefined:$b = -2$ Zero:$b = 2$ or $b = 3$
13. $a \geq 4$
14. $x = -\frac{3}{2}$ or $x = 1$
15. a) $x < 3$ or $x \geq 7$:
    b) $x < 1$ or $x > 5$:
    c) $3 < x < 7$:
    d) $x < -1$ or $x > 3$
    e) $0{,}5 < x < 2{,}5$
    f) $x \leq -3$ or $0 < x \leq \frac{5}{2}$
    g) $x < \frac{2}{3}$
    h) $-1 \leq x < 0$ or $x \geq 3$
    i) $-4 \leq x \leq 1$
    j) $2\frac{1}{2} \leq x < 3$
16. a) $x = \pm\sqrt{3}$ and $y = \pm 2\sqrt{3}$
    b) $a = -3$ and $b = -1$ or $a = 12$ and $b = 4$
    c) $x = -5$ and $y = 0$ or $x = 2$ and $y = 14$
    d) $p = \frac{5}{3}$ and $q = \frac{2}{9}$ or $p = -1$ and $q = \frac{-2}{3}$
    e) $b = \frac{3\pm\sqrt{5}}{2}$ and $a = \frac{7\pm 3\sqrt{5}}{2}$
    f) $b = \frac{-10\pm\sqrt{140}}{4}$ and $a = \frac{-12\pm\sqrt{140}}{2}$
    g) $x = 3{,}4$ and $y = 5{,}4$ or $x = 3$ and $y = 5$
    h) $b = -1{,}4$ and $a = 23{,}6$ or
    $b = 3$ and $a = 6$
17. a)

b)

18. 35 m
20. a) $y = -\frac{5}{4}$ or $y = -9$
    b) $x = -\frac{9}{4}$ or $x = 1$
    c) $p = -\frac{8}{7}$ or $p = -\frac{4}{3}$
    d) $y = -\frac{1}{4}$ or $y = \frac{1}{2}$
    e) $y = -\frac{2}{9}$ or $y = -1$
    f) $y = \frac{7}{3}$ or $y = -\frac{1}{2}$
    g) $y = \frac{9}{4}$ or $y = -\frac{9}{4}$
    h) $y = \frac{8}{3}$ or $y = -6$
    i) $y = \frac{9}{5}$ or $y = -7$
    j) $x = \pm 4$
    k) $y = \pm 7$
21. $k = 76$ and $\frac{4}{9}$
22. $x = 3$ or $x = -2$ and $y = \frac{1\pm\sqrt{-7}}{2}$
23. $x = 4$ or $x = -1$
24. $y = \frac{3}{2}, y = \frac{1}{2}$ and $p = \frac{9}{2}, p = \frac{7}{2}$
25. $\frac{69}{4}$
26. 7
27. $\frac{2\pm\sqrt{12}}{2}$
28. $t = \frac{1}{2}, t = 1$ or $t = \frac{3\pm\sqrt{33}}{4}$

Exercise 3 – 1: Linear sequences

1. -19; -35; -51
2. a) -19
   b) $T_2 = 15$; $T_4 = 33$
3. a) $T_n = 10 + 3n$; $T_{10} = 40$; $T_{15} = 55$; $T_{30} = 100$
   b) $T_n = 12 + 6n$; $T_{10} = 72$; $T_{15} = 102$; $T_{30} = 192$
   c) $T_n = -5 - 5n$; $T_{10} = -55$; $T_{15} = -80$; $T_{30} = -155$
4. $T_9 = 36$
5. a) 44; 66; 121

Exercise 3 – 2: Quadratic sequences

1. a) 10
   b) 2
   c) 2
   d) -2
   e) 2
   f) -4
   g) 4
   h) -2
   i) $6a$
   j) 6
   k) $2t$
2. a) $T_4 = 53$
   b) $T_2 = 30$
   c) $T_1 = 17$
   d) $T_2 = -3$
   e) $T_4 = 63$
   f) $T_1 = 2$
3. a) 3; 9; 17; 27
   b) -6; -9; -14; -21
   c) 1; 8; 21; 40
   d) 0; -5; -14; -27

Exercise 3 – 3: Quadratic sequences

1. a) 1
   b) 2
   c) 4
   d) 8
   e) -2
2. 12; 30; 58; 96; 144
3. $T_9 = 379$
4. $n = 4$
5. a) $T_5 = 84$; $T_6 = 111$
   b) $T_n = 2n^2 + 5n + 9$

Exercise 3 – 4: End of chapter exercises

1. $-4; 9; 16; 25; 36$
2. a) Quadratic sequence
   b) Quadratic sequence
   c) Quadratic sequence
   d) Quadratic sequence
   e) Quadratic sequence
   f) Quadratic sequence
   g) Linear sequence
   h) Linear sequence
   i) Quadratic sequence
   j) Quadratic sequence
   k) Quadratic sequence
   l) Linear sequence
   m) Quadratic sequence
3. $x = 31$
4. $n = 11$
5. $T_11 = 363$
6. $n = 9$
7. $T_5 = 114$
8. $n = 8$
9. a) $T_5 = 19$; $T_n = 4n - 1$; $T_{10} = 39$
   b) $T_5 = -3$; $T_n = 22 - 5n$; $T_{10} = -28$
   c) $T_5 = 2\frac{1}{2}$; $T_n = \frac{1}{2}n$; $T_{10} = 5$
   d) $T_5 = a + 4b$; $T_n = a - b + bn$; $T_{10} = a + 9b$
   e) $T_5 = -7$; $T_n = 3 - 2n$; $T_{10} = -17$
10. a) $T_n = n^2 + 3$; $T_{100} = 10\ 003$
    b) $T_n = 6n - 4$; $T_{100} = 596$
    c) $T_n = 2n^2 + 5$; $T_{100} = 20\ 005$
    d) $T_n = 3n^2 + 2$; $T_{100} = 30\ 002$
11. a) $2; 5; 8; 11; 14$
    b) Constant difference, $d = 3$
    c) Yes
12. a) $T_n = 4n - 19$
    b) $n = 48$
13. a) Incorrect
    b) Correct
14. c) Linear
15. b) Linear
    d) Quadratic
    e) $T_n = \frac{1}{2}n^2 + \frac{3}{2}n + 1$
    f) $T_{21} = 253$
    g) 31 cm
16. a) -1
    b) 7
17. b) 2
    c) $T_n = n^2 - n$
    d) 210
    e) 25
18. $4; 14; 34; 64; 104; 154$

4 Analytical geometry

Exercise 4 – 1: Revision

1. a) $2\sqrt{26}$units
   b) 7 units
   c) $x + 1$units
2. $p = 6$ or $p = 2$
3. a) $-\frac{1}{2}$
   b) 3
5. 2
6. a) $(1; 2)$
   b) $\{\frac{-1}{2}; \frac{-1}{2})$
7. $B(4; 2)$
8. a) $y = -4x + 3$ and $y = -4x + 19$
   c) $AD = \sqrt{17}$units and $BC = \sqrt{17}$units
   d) $y = \frac{4}{3}x - \frac{7}{3}$
   e) Parallelogram (one opposite side equal and parallel)
9. $N(0; 3)$
10. a) $PQ = \sqrt{20}$ and $SR = \sqrt{20}$
    b) $M(\frac{3}{2}; 1)$
    d) PS: $y = -\frac{2}{5}x - \frac{1}{5}$ and SR: $y = \frac{1}{2}x - 2$
    e) No
    f) Parallelogram

Exercise 4 – 2: The two-point form of the straight line equation

1. $y = \frac{2}{3}x + 5$
2. $y = -3x + \frac{1}{4}$
3. $y = x + 3$
4. $y = 2x - 1$
5. $y = -5$
6. $y = \frac{3}{4}x + 3$
7. $y = -x + (s + t)$
8. $y = 5x + 2$
9. $y = \frac{q}{p}x - q$

Exercise 4 – 3: Gradient–point form of a straight line equation

1. $y = \frac{2}{3}x + 4$
2. $y = -x - 2$
3. $y = -\frac{1}{3}x$
4. $y = 11$
5. $y = -2x + 7$
6. $x = -\frac{3}{2}$
7. $y = -\frac{4}{5}x + 1$
8. $x = 4$
9. $y = 3ax + b$

Exercise 4 – 4: The gradient–intercept form of a straight line equation

1. $y = 2x + 3$
2. $y = 4x - 4$
3. $y = -x - 1$
4. $y = -\frac{3}{7}x$
5. $y = \frac{1}{2}x - \frac{1}{5}$
6. $y = 2x - 2$
7. $y = -\frac{3}{2}$
8. $y = 3x + 4$
9. $y = -5x$

Exercise 4 – 5: Angle of inclination

1. a) 1,7
   b) -1
   c) 0
   d) 1,4
   e) Undefined
   f) 1
   g) $-0{,}8$
   h) 0
   i) 3,7
2. a) 36,8°
   b) 26,6°
   c) 45°
   d) Horizontal line
   e) 18,4°
   f) Vertical line
   g) 71,6°
   h) 30°

Exercise 4 – 6: Inclination of a straight line

1. a) 38,7°
   b) 135°
   c) 80°
   d) 80°
   e) 102,5°
   f) 45°
   g) 56,3°
   h) 63,4°
   i) 161,6°
   j) Gradient undefined
2. 85,2°
3. 90°
4. 81,8°

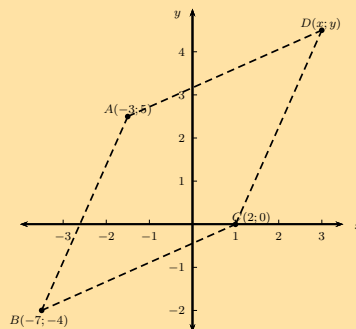
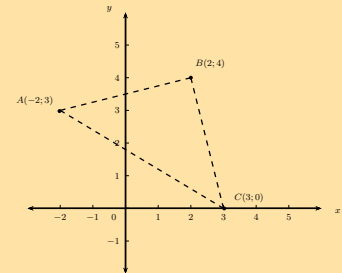
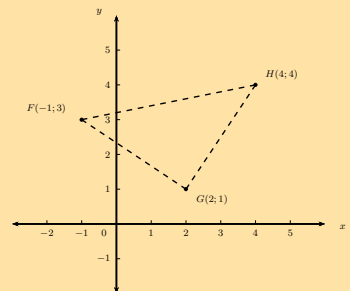
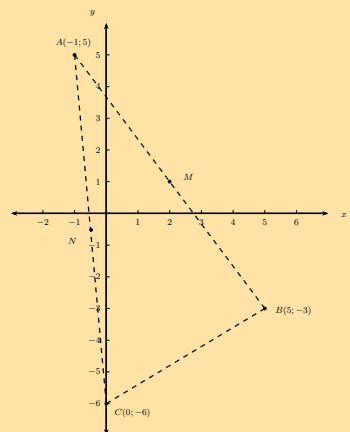
Exercise 4 – 7: Parallel lines

1. a) Parallel
   b) Parallel
   c) Parallel
   d) Not parallel
   e) Parallel
   f) Parallel
2. $y = -2x - 3$
3. $y = 3x$
4. $y = \frac{3}{2}x + 1$
5. $y = -\frac{7}{10}x - 1$

Exercise 4 – 8: Perpendicular lines

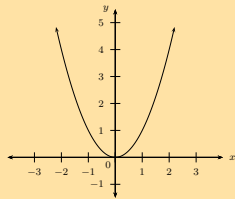
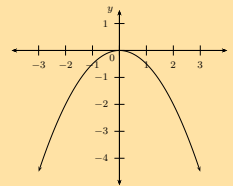
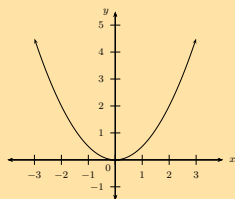
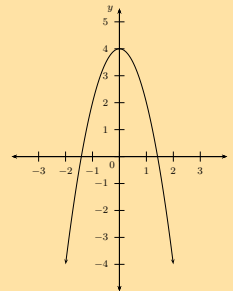
1. a) Perpendicular
   b) Not perpendicular
   c) Perpendicular
   d) Perpendicular
   e) Perpendicular
   f) Not perpendicular
   g) Not perpendicular
2. $y = \frac{1}{2}x - 3$
3. $y = -5x + 3$
4. $y = -x + 2$
5. $x = -2$

Exercise 4 – 9: End of chapter exercises

1. a) $y = \frac{1}{2}x + \frac{7}{2}$
   b) $y = -x + 4$
   c) $y = \frac{1}{2}x + 4$
   d) $y = 2x + 4$
   e) $y = 3x$
2. a) $\theta = 63{,}4°$
   b) $\theta = 18{,}4°$
   c) $\theta = 36{,}9°$
   d) $\theta = 146{,}3°$
   e) $\theta = 161{,}6°$
3. a) $y = -2x + 7$
   b) $\left(\frac{7}{2}; 0\right)$
   c) $\theta = 116{,}6°$
   d) $m = \frac{1}{2}$
   e) $Q\hat{P}R = 90°$
   f) $y = -2x$
   g) $\left(\frac{1}{2}; -\frac{3}{2}\right)$
   h) $y = -2x - \frac{1}{2}$
4. a)

   b) $D\,(6; 9)$
5. a) $(-1; -2)$
   b) $(8; 3)$
   c) $x = -1$
   d) $MN = 5$ units
   e) $M\hat{N}P = 21{,}8°$
   f) $y = \frac{5}{2}x + \frac{11}{2}$
6. a)

   c) $y = \frac{1}{4}x + \frac{7}{2}$
   d) $D(-1; -1)$
   e) $E\left(\frac{5}{2}; 2\right)$
7. a) $y = \frac{3}{2}x + 2$
   b) $T\hat{S}V = 49{,}6°$
8. a)

   c) $y = -5x + 11$
   d) Yes
   e) $y = \frac{3}{2}x + \frac{9}{2}$
9. a)

Exercise 5 – 1: Revision

1. a)

b)

c)

d)

Exercise 5 – 2: Domain and range

1. $\{x : x \in \mathbb{R}\}; \{y : y \geq -1, y \in \mathbb{R}\}$
2. $\{x : x \in \mathbb{R}\}; \{y : y \leq 4, y \in \mathbb{R}\}$
3. $\{x : x \in \mathbb{R}\}; \{y : y \geq 0, y \in \mathbb{R}\}$
4. $\{x : x \in \mathbb{R}\}; \{y : y \leq 0, y \in \mathbb{R}\}$
5. $\{x : x \in \mathbb{R}\}; \{y : y \leq 2, y \in \mathbb{R}\}$

Exercise 5 – 3: Intercepts

1. $(0; 15)$ and $(-5; 0); (-3; 0)$
2. $(0; 16)$ and $(4; 0)$
3. $(0; -3)$ and $(1; 0); (3; 0)$
4. $(0; 35)$ and $(-\frac{7}{2}; 0); (-\frac{5}{2}; 0)$
5. $(0; 37)$ and no x-intercepts
6. $(0; -4)$ and $(-0{,}85; 0); (-2{,}35; 0)$

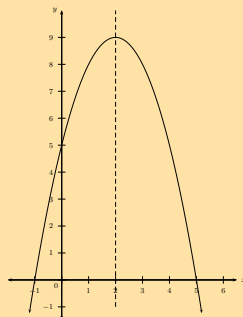
Exercise 5 – 4: Turning points

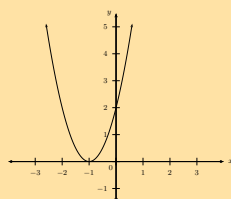
1. $(3; -1)$
2. $(2; 1)$
3. $(-2; -1)$
4. $(-\frac{1}{2}; \frac{1}{2})$
5. $(1; 21)$
6. $(-1; -6)$

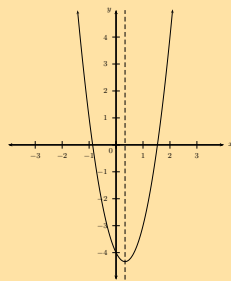
Exercise 5 – 5: Axis of symmetry

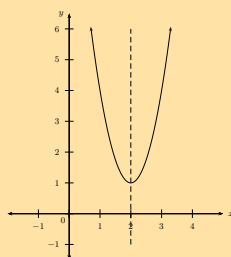
1. a) Axis of symmetry: $x = \frac{5}{4}$
   b) Axis of symmetry: $x = 2$
   c) Axis of symmetry: $x = 2$
2. $y = ax^2 + q$

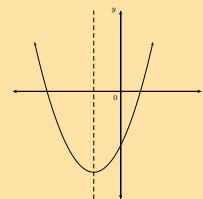
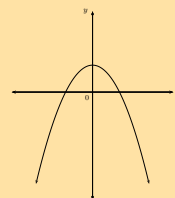
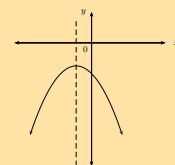
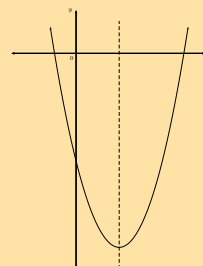
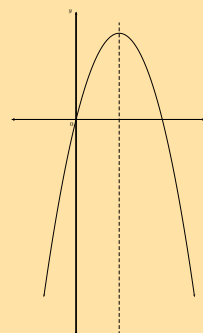
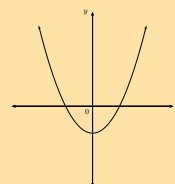
Exercise 5 – 6: Sketching parabolas

1. a)

Intercepts: $(-1;0), (5;0), (0;5)$
Turning point: $(2;9)$
Axes of symmetry: $x = 2$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \leq 9, y \in \mathbb{R}\}$

b)

Intercepts: $(-1;0), (0;2)$
Turning point: $(-1;0)$
Axes of symmetry: $x = -1$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 0, y \in \mathbb{R}\}$

c)

Intercepts: $(-0{,}87;0), (1{,}54;0), (0;-4)$
Turning point: $(0{,}33;-4{,}33)$
Axes of symmetry: $x = -0{,}33$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 4{,}33, y \in \mathbb{R}\}$

d)

Intercepts: $(0;13)$ Turning point: $(2;1)$
Axes of symmetry: $x = 2$
Domain: $\{x : x \in \mathbb{R}\}$
Range: $\{y : y \geq 1, y \in \mathbb{R}\}$

3. a)

b)

c)

d)

e)

f)

4. a) $y_{\text{shifted}} = 2x^2 + 16x + 32$
   b) $y_{\text{shifted}} = -x^2 - 2x$
   c) $y_{\text{shifted}} = 3x^2 - 16x + 22$

Exercise 5 – 7: Finding the equation

1. $y = -3(x+1)^2 + 6$ or $y = -3x^2 - 6x + 3$
2. $y = \frac{1}{2}x^2 - \frac{5}{2}x$
3. $y = \frac{2}{3}(x+2)^2$
4. $y = -x^2 + 3x + 4$

Exercise 5 – 8:

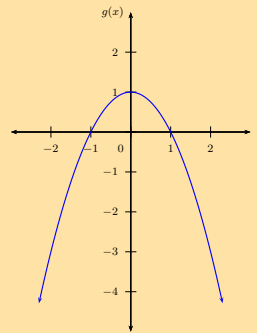
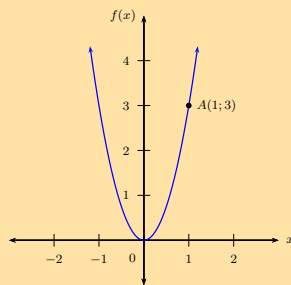
1. a) 11

2. a)

   b) 6

   c) $y = 6x - 3$

3. a)

   b) 1

   c) 4

   d) 0

Exercise 5 – 10: Domain and range

1. $\{x : x \in \mathbb{R}, x \neq 0\}; \{y : y \in \mathbb{R}, y \neq 1\}$
2. $\{x : x \in \mathbb{R}, x \neq 8\}; \{y : y \in \mathbb{R}, y \neq 4\}$
3. $\{x : x \in \mathbb{R}, x \neq -1\}; \{y : y \in \mathbb{R}, y \neq -3\}$
4. $\{x : x \in \mathbb{R}, x \neq 5\}; \{y : y \in \mathbb{R}, y \neq 3\}$
5. $\{x : x \in \mathbb{R}, x \neq -2\}; \{y : y \in \mathbb{R}, y \neq 2\}$

Exercise 5 – 11: Intercepts

1. $(0; -1\frac{3}{4})$ and $\left\{-3\frac{1}{2}; 0\right)$
2. $\left\{\frac{5}{2}; 0\right)$
3. $(0; 1)$ and $\left\{\frac{1}{3}; 0\right)$
4. $\left\{0; \frac{3}{2}\right)$ and $\left\{\frac{1}{3}; 0\right)$
5. $(0; 2)$ and $(8; 0)$

Exercise 5 – 12: Asymptotes

1. $y = -2$ and $x = -4$
2. $y = 0$ and $x = 0$
3. $y = 1$ and $x = 2$
4. $y = -8$ and $x = 0$
5. $y = 0$ and $x = 2$

Exercise 5 – 13: Axes of symmetry

1. a) For $f(x)$: $(0;0)$; $y_1 = x$ and $y_2 = -x$
   For $g(x)$: $(0;1)$; $y_1 = x + 1$ and $y_2 = -x + 1$
   b) For $f(x)$: $(0;0)$; $y_1 = x$ and $y_2 = -x$
   For $g(x)$: $(-1;0)$; $y_1 = x + 1$ and $y_2 = -x - 1$
   c) For $f(x)$: $(0;0)$; $y_1 = x$ and $y_2 = -x$
   For $g(x)$: $(1;-1)$; $y_1 = x - 2$ and $y_2 = -x$
2. $k(x) = \frac{5}{x+1} + 2$

Exercise 5 – 14: Sketching graphs

1. a) Asymptotes: $x = 0$; $y = 2$
   Intercepts: $\left(-\frac{1}{2};0\right)$
   Axes of symmetry: $y = x + 2$ and $y = -x + 2$
   Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$
   Range: $\{y : y \in \mathbb{R}, y \neq 2\}$
   b) Asymptotes: $x = -4$; $y = -2$
   Intercepts: $\left(-3\frac{1}{2};0\right)$ and $\left(0;-1\frac{3}{4}\right)$
   Axes of symmetry: $y = x + 2$ and $y = -x - 6$
   Domain: $\{x : x \in \mathbb{R}, x \neq -4\}$
   Range: $\{y : y \in \mathbb{R}, y \neq -2\}$
   c) Asymptotes: $x = -1$; $y = 3$
   Intercepts: $\left(-\frac{2}{3};0\right)$ and $(0;2)$
   Axes of symmetry: $y = x + 4$ and $y = -x + 2$
   Domain: $\{x : x \in \mathbb{R}, x \neq -1\}$
   Range: $\{y : y \in \mathbb{R}, y \neq 3\}$
   d) Asymptotes: $x = -2\frac{1}{2}$; $y = -2$
   Intercepts: $(0;0)$
   Axes of symmetry: $y = x - 4\frac{1}{2}$ and $y = -x + \frac{1}{2}$
   Domain: $\{x : x \in \mathbb{R}, x \neq -4\}$
   Range: $\{y : y \in \mathbb{R}, y \neq -2\}$
   e) Asymptotes: $x = 8$; $y = 4$
   Intercepts: $(6;0)$ and $(0;3)$
   Axes of symmetry: $y = x - 4$ and $y = -x + 12$
   Domain: $\{x : x \in \mathbb{R}, x \neq 8\}$
   Range: $\{y : y \in \mathbb{R}, y \neq 4\}$
2. $y = \frac{1}{x+2} - 1$
3. $y = -\frac{4}{x} + 2$
4. a)
   b) Average gradient $= 1$
   c) Average gradient $= 12$

Exercise 5 – 16: Domain and range

1. $\{x : x \in \mathbb{R}\}; \{y : y > 0, y \in \mathbb{R}\}$
2. $\{x : x \in \mathbb{R}\}; \{y : y < 1, y \in \mathbb{R}\}$
3. $\{x : x \in \mathbb{R}\}; \{y : y > -3, y \in \mathbb{R}\}$
4. $\{x : x \in \mathbb{R}\}; \{y : y > n, y \in \mathbb{R}\}$
5. $\{x : x \in \mathbb{R}\}; \{y : y > 2, y \in \mathbb{R}\}$

Exercise 5 – 17: Intercepts

1. $(0;-6)$ and $(2;0)$
2. $(0;-17\frac{1}{3})$ and $(3;0)$
3. $(0;-20)$ and $(-1;0)$
4. $(0;\frac{15}{16})$ and $(-2;0)$

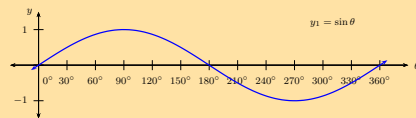
Exercise 5 – 18: Asymptote

1. $y = 0$
2. $y = 1$
3. $y = -\frac{2}{3}$
4. $y = -2$
5. $y = -2$

Exercise 5 – 19: Mixed exercises

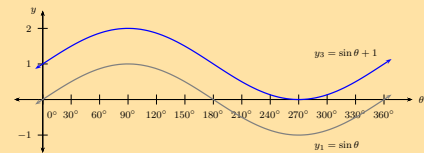
1. b) i. $y = \frac{3}{x} + 3$
   ii. $y = \frac{3}{x-3}$
   iii. $y = -\frac{3}{x}$
   iv. $y = \frac{3}{x} - \frac{1}{4}$
   v. $y = \frac{3}{x} + 4$
   vi. $y = \frac{3}{x+2} - 1$
2. a) $M(-2; 2)$
   b) $g(x) = \frac{-4}{x}$
   c) $f(x) = 2(x + 1)^2$
   d) $-2 < x < 0$
   e) Range: $\{y : y \in \mathbb{R}, y \geq 0\}$
3. a) For $k(x)$:
   Intercepts: $(-2; 0), (1; 0)$ and $(0; -4)$
   Turning point: $\{-\frac{1}{2}; -4\frac{1}{2})$
   Asymptote: none
   For $h(x)$:
   Intercepts: $(1{,}41; 0)$
   Turning point: none
   Asymptote: $y = 0$
6. a) $f(x) = -\frac{3}{4}(x - 2)^2 + 3$;
   Axes of symmetry: $x = 2$;
   Domain: $\{x : x \in \mathbb{R}\}$;
   Range: $\{y : y \in \mathbb{R}, y \leq 3\}$
   b) $g(x) = \frac{1}{4}x^2 - 2$;
   Axes of symmetry: $x = 0$;
   Domain: $\{x : x \in \mathbb{R}\}$;
   Range: $\{y : y \in \mathbb{R}, y \geq -2\}$; $h(x) = \frac{2}{x}$;
   Axes of symmetry: $y = x$
   Domain: $\{x : x \in \mathbb{R}, x < 0\}$;
   Range: $\{y : y \in \mathbb{R}, y < 0\}$;
   c) $k(x) = \{\frac{1}{2})^x + \frac{1}{2}$;
   Domain: $\{x : x \in \mathbb{R}\}$;
   Range: $y : y \in \mathbb{R}, y > \frac{1}{2}\}$
7. b) $p = 9$
   c) Average gradient $= -2\frac{8}{9}$
   d) $y = \{\frac{1}{3})^{x+2} - 2$
8. a) $f(x) = 2^x - \frac{3}{2}$ and $g(x) = -\frac{1}{4}x - \frac{1}{2}$
   b) $h(x) = -\frac{3}{x+2} + 1$
9. a) $AO = 2$ units $OB = 5$ units
   $OC = 10$ units $DE = 12{,}25$ units
   b) $DE = 12\frac{1}{4}$
   c) $h(x) = -2x + 10$
   d) $\{x : x \in \mathbb{R}, x < -2$ and $x > 5\}$
   e) $\{x : x \in \mathbb{R}, 0 \leq x \leq 5\}$
   f) 5,25 units

Exercise 5 – 20: Revision

1.

$$\begin{aligned}
\text{Period: } &= 360^\circ \\
\text{Amplitude: } &= 1 \\
\text{Domain: } &= [0^\circ; 360^\circ] \\
\text{Range: } &= [-1; 1] \\
x\text{-intercepts: } &= (0^\circ; 0); (180^\circ; 0); (360^\circ; 0) \\
y\text{-intercepts: } &= (0^\circ; 0) \\
\text{Max. turning point: } &= (90^\circ; 1) \\
\text{Min. turning point: } &= (270^\circ; -1)
\end{aligned}$$

2.

$$\begin{aligned}
\text{Period: } & 360^\circ \\
\text{Amplitude: } & 2 \\
\text{Domain: } & [0^\circ; 360^\circ] \\
\text{Range: } & [-2; 2] \\
x\text{-intercepts: } & (0^\circ; 0); (180^\circ; 0); (360^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; 0) \\
\text{Max. turning point: } & (270^\circ; 2) \\
\text{Min. turning point: } & (90^\circ; -2)
\end{aligned}$$

3.

$$\begin{aligned}
\text{Period: } & 360^\circ \\
\text{Amplitude: } & 1 \\
\text{Domain: } & [0^\circ; 360^\circ] \\
\text{Range: } & [0; 2] \\
x\text{-intercepts: } & (270^\circ; 0) \\
y\text{-intercepts: } & (0^\circ; 1) \\
\text{Max. turning point: } & (90^\circ; 2) \\
\text{Min. turning point: } & (270^\circ; 0)
\end{aligned}$$

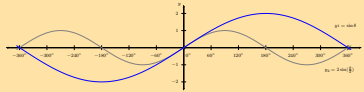
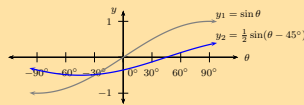
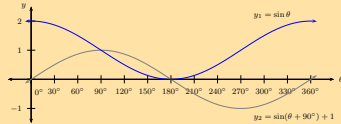
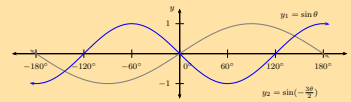
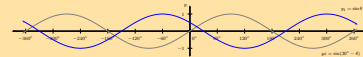
4.

$$\begin{aligned}
\text{Period: } & 360^\circ \\
\text{Amplitude: } & \frac{1}{2} \\
\text{Domain: } & [0^\circ; 360^\circ] \\
\text{Range: } & [-\frac{1}{2}; -\frac{3}{2}] \\
x\text{-intercepts: } & \text{none} \\
y\text{-intercepts: } & (0^\circ; -\frac{1}{2}) \\
\text{Max. turning point: } & (90^\circ; \frac{1}{2}) \\
\text{Min. turning point: } & (270^\circ; -\frac{3}{2})
\end{aligned}$$

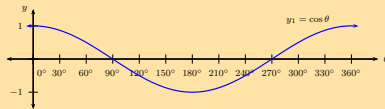
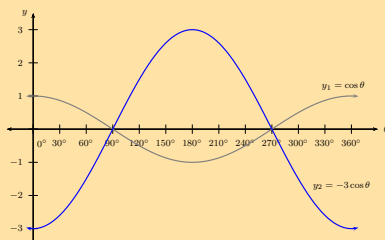
Exercise 5 – 21: Sine functions of the form $y = \sin k\theta$

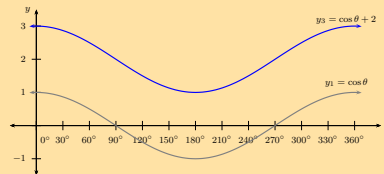
2. a) $k = 2$ b) $k = -\frac{3}{4}$

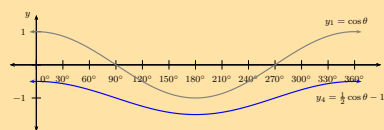
Exercise 5 – 23: The sine function

1. a)

b)

c)

d)

e)

2. $a = 2$; $p = 90°$ $\therefore y = 2\sin(\theta + 90°)$ and $y = 2\cos\theta$

Exercise 5 – 24: Revision

1.

2.

Period: $360°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[-3; 3]$
x-intercepts: none
y-intercepts: $(0°; 3)$
Max. turning point: $(0°; 3)$; $(360°; 3)$
Min. turning point: $(180°; -3)$

3.

Period: $360°$
Amplitude: 1
Domain: $[0°; 360°]$
Range: $[1; 3]$
x-intercepts: none
y-intercepts: $(0°; 3)$
Max. turning point: $(0°; 3)$; $(360°; 3)$
Min. turning point: $(180°; 1)$

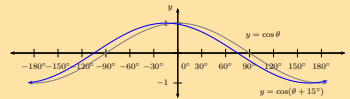
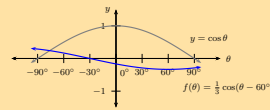
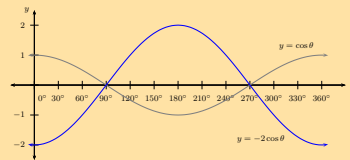
4.

Period: $360°$
Amplitude: $\frac{1}{2}$
Domain: $[0°; 360°]$
Range: $[-\frac{1}{2}; -\frac{3}{2}]$
x-intercepts: none
y-intercepts: $(0°; -\frac{1}{2})$
Max. turning point: $(0°; -\frac{1}{2})$; $(360°; -\frac{1}{2})$
Min. turning point: $(180°; -\frac{3}{2})$

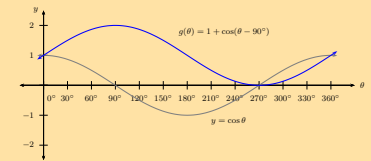
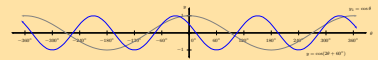
Exercise 5 – 25: Cosine functions of the form $y = \cos k\theta$

2. a) $k = \frac{3}{2}$

b) $k = \frac{2}{3}$

Exercise 5 – 27: The cosine function

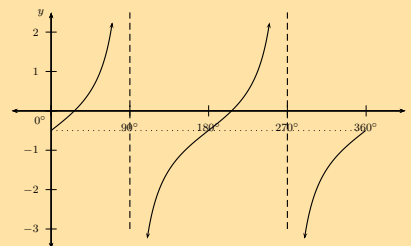
1. a)

b)

c)

d)

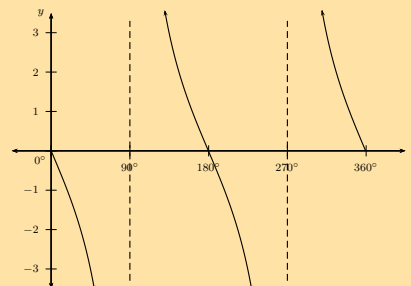
e)

f)

2. a) $a = -1$

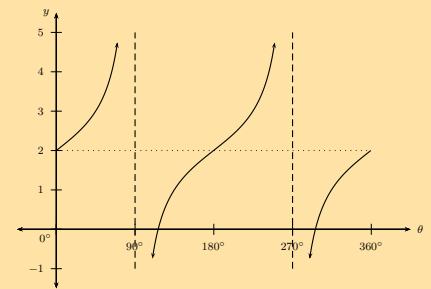
b) $p = -180°$

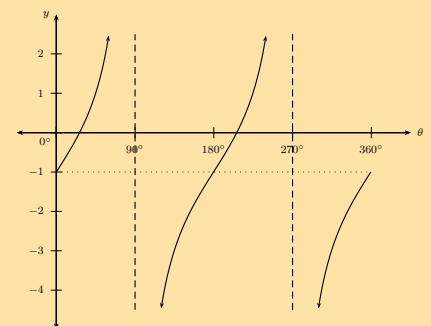
c) $\cos(\theta - 180°) = -\cos\theta$

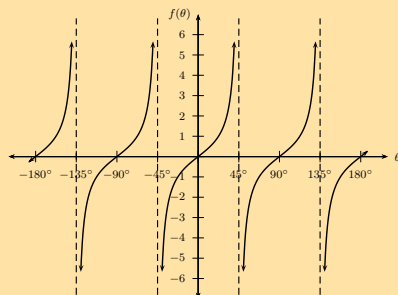
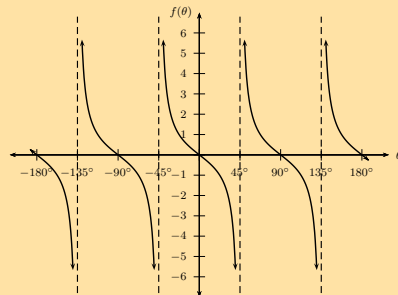
Exercise 5 – 28: Revision

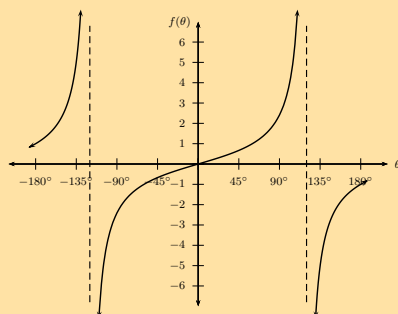
1.

Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(26{,}6°; 0)$; $(206{,}6°; 0)$
y-intercepts: $(0°; -\frac{1}{2})$
Asymptotes: $90°; 270°$

2.

Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(0°; 0)$; $(180°; 0)$; $(360°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $90°; 270°$

3.

Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(116{,}6°; 0)$; $(296{,}6°; 0)$
y-intercepts: $(0°; 2)$
Asymptotes: $90°; 270°$

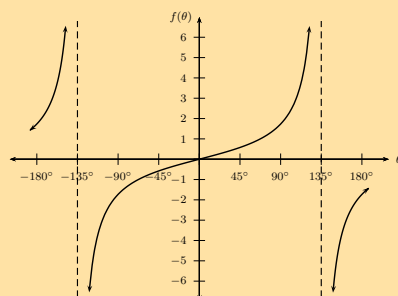
4.

Period: $180°$
Domain: $[0°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(26{,}6°; 0)$; $(206{,}6°; 0)$
y-intercepts: $(0°; -1)$
Asymptotes: $90°; 270°$

Exercise 5 – 29: Tangent functions of the form $y = \tan k\theta$

1.

Period: $90°$
Domain: $[-180°; 180°]$ Range: $[-\infty; \infty]$
x-intercepts: $(-180°; 0)$; $(-90°; 0)$; $(0°; 0)$; $(90°; 0)$; $(180°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-135°$; $-45°$; $45°$; $135°$

2.

Period: $240°$
Domain: $[-180°; 180°]$
Range: $[-\infty; \infty]$
x-intercepts: $(0°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-120°$; $120°$

3.

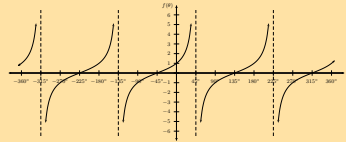
Period: $90°$
Domain: $[-180°; 180°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-180°; 0)$; $(-90°; 0)$; $(0°; 0)$; $(90°; 0)$; $(180°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-135°$; $-45°$; $45°$; $135°$

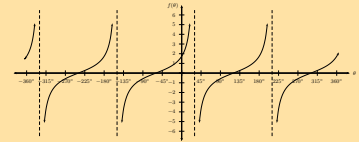
4.

Period: $270°$
Domain: $[-180°; 180°]$
Range: $[-\infty; \infty]$
x-intercepts: $(0°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-135°$; $135°$

Exercise 5 – 30: Tangent functions of the form $y = \tan(\theta + p)$

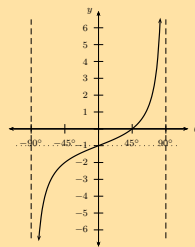
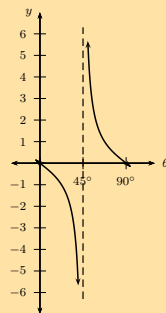
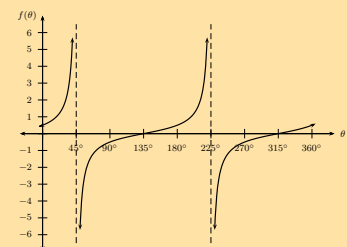
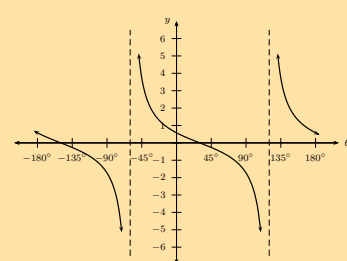
1.

Period: 180°
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-225°; 0)$; $(-45°; 0)$; $(135°; 0)$; $(315°; 0)$
y-intercepts: $(0°; 0)$
Asymptotes: $-315°$; $-135°$; $45°$; $225°$

2.

Period: 180°
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-330°; 0)$; $(-150°; 0)$; $(30°; 0)$; $(210°; 0)$
y-intercepts: $(0°; -0{,}58)$
Asymptotes: $-315°$; $-135°$; $45°$; $225°$

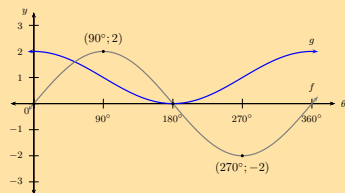
3.

Period: 180°
Domain: $[-360°; 360°]$
Range: $[-\infty; \infty]$
x-intercepts: $(-240°; 0)$; $(-60°; 0)$; $(120°; 0)$; $(300°; 0)$
y-intercepts: $(0°; 1.73)$
Asymptotes: $-330°$; $-150°$; $30°$; $210°$

Exercise 5 – 31: The tangent function

1. a)

b)

c)

d)

2. $a = -1$; $k = \frac{1}{2}$

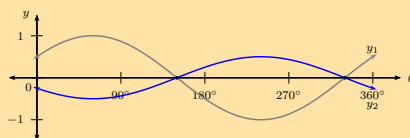
Exercise 5 – 32: Mixed exercises

1. a) $f(\theta) = \frac{3}{2}\sin 2\theta$ and $g(\theta) = -\frac{3}{2}\tan\theta$
   b) $f(\theta) = -2\sin\theta$ and $g(\theta) = 2\cos(\theta + 360°$
   c) $y = 3\tan\frac{\theta}{2}$
   d) $y = y = 2\cos\theta + 2$

2. a)

   b) 360°

   c) 1
   d) At $\theta = 180°$

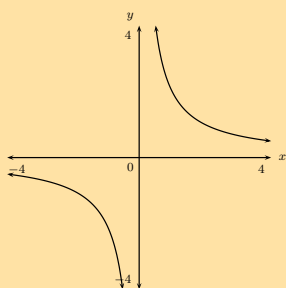
3. a) $a = 2$, $b = -1$ and $c = 240°$
   b) 180°
   c) $\theta = 60°; 300°$
   d) $y = -\tan(\theta - 45°)$

4.

Exercise 5 – 33: End of chapter exercises

2. $a = -2; k = -1$

4. a) $y = x + 2^2 + 2$
   b) $y = x - 1^2 + 5$

5. $(-1; 0)$

6.

$y = \frac{2}{x-3} - 1$

7. $y = \frac{1}{(x-1)} + 2$

9. a) $a = -1$
   b) $f(-15) = 0{,}99997$
   c) $x = -1$
   d) $h(x) = -2^{(x-2)} + 1$

10. a) $a = 256$
    b) $f(x) = 256\left\{\frac{3}{4}\right)^x$
    c) $f(13) = 6{,}08$

11. a)

    b)

    d)

    e)

    f)

Exercise 6 – 1: Revision

1. a) True
   b) True
   c) False
   d) True
2. a) 50,2°
   b) 40,5°
   c) 26,6°
   d) 109,8°
   e) No solution
   f) 17,7°
   g) 69,4°
3. a) 17,3 cm
   b) 10 cm
   c) 64,8°
4. a) 10 cm
   b) 5,2 cm and 19,3 cm
   c) 50 cm^2
5. a) 2
   b) 0
   c) $-1\frac{1}{2}$
   d) 1
   e) 1
6. a) 60°
   b) $\frac{1}{2}$
   c) 1
7. No

Exercise 6 – 2: Trigonometric identities

1. a) $\cos \alpha$
   b) $\tan^2 \theta$
   c) $\cos^2 \theta$
   d) 0

Exercise 6 – 3: Reduction formulae for function values of $180° \pm \theta$

1. a) $\frac{\sqrt{3}}{3}$
   b) $\frac{1}{8}$
   c) 1
2. a) $\frac{1-\cos^2 \theta}{\cos \theta}$
   b) -1
3. a) $2t$
   b) $-\frac{1}{t}$

Exercise 6 – 4: Using reduction formula

1. a) $-\tan \theta$
   b) 1
   c) 1
2. $-\cos \beta$
3. a) $\frac{1}{\sqrt{3}}$
   b) 2
   c) 2
   d) $-\frac{3}{2}$
   e) $-\frac{4\sqrt{3}}{5}$
5. a) $-t$
   b) $1-t^2$
   c) $\pm\frac{t}{\sqrt{1-t^2}}$

Exercise 6 – 5: Co-functions

1. a) $\cos \theta$
   b) $\frac{3}{2}$
2. a) p
   b) $\sqrt{1-p^2}$
   c) $-\frac{p}{\sqrt{1-p^2}}$
   d) p

Exercise 6 – 6: Reduction formulae

1.
   a) $\sin^2\theta$
   b) $\cos^2\theta$
   c) i. 1
      ii. $\tan^2\theta$

2.
   a) $\sin 17°$
   b) $\cos 33°$
   c) $\tan 68°$
   d) $-\cos 33°$

3.
   a) $\sqrt{3}$
   b) $\frac{\sqrt{3}}{2}$
   c) $\frac{1}{4}$
   d) 1

Exercise 6 – 7: Solving trigonometric equations

1.
   a) $\alpha = 60°; 300°$
   b) $\alpha = 220{,}5°; 319{,}5°$
   c) $\alpha = 79{,}2°; 259{,}2°$
   d) $\alpha = 200{,}1°; 339{,}9°$
   e) $\alpha = 36{,}9°; 143{,}1°$
   f) $\alpha = 109{,}7°; 289{,}7°$

2.
   a) $\theta = -323{,}1°; -216{,}9°; 36{,}9°; 143{,}1°$
   b) $\theta = -221{,}4°; -138{,}6°; 138{,}6°; 221{,}4°$
   c) $\theta = -278{,}5°; -98{,}5°; 81{,}5°; 261{,}5°$
   d) $\theta = -90°; 270°$
   e) $\theta = -293{,}6°; -66{,}4°; 66{,}4°; 293{,}6°$

Exercise 6 – 8: General solution

1.
   a) $\theta = -128{,}36°; -101{,}64°; 51{,}64°$
   b) $\theta = -80{,}45°; -9{,}54°; 99{,}55°; 170{,}46°$
   c) $\theta = -53{,}27°; 126{,}73°$
   d) $\alpha = 0°$
   e) $\theta = -180°; 0°; 180°$
   f) $\theta = -180°; 180°$
   g) $\theta = 84°$
   h) $\theta = -120°; 120°$
   i) $\theta = -60°; -30°; 120°; 150°$

2.
   a) $\theta = -20° + n\ .\ 360°$
   b) $\alpha = 30° + n\ .\ 120°$
   c) $\beta = 10{,}25° + n\ .\ 45°$ or
      $\beta = 55{,}25° + n\ .\ 45°$
   d) $\alpha = 70° + n\ .\ 360°$ or
      $\alpha = 340° + n\ .\ 360°$
   e) $\theta = 140° + n\ .\ 240°$ or
      $\theta = 220° + n\ .\ 240°$
   f) $\beta = 15° + n\ .\ 180°$

Exercise 6 – 9: Solving trigonometric equations

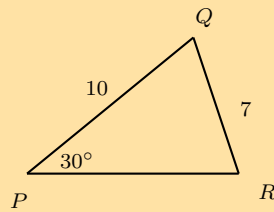
1.
   a) $\theta = 45° + k\ .\ 180°$ or
      $\theta = 135° + k\ .\ 180°$
   b) $\alpha = 50° + k\ .\ 360°$ or
      $\alpha = 110° + k\ .\ 360°$
   c) $\theta = 60° + k\ .\ 720°$ or
      $\theta = 660° + k\ .\ 720°$
   d) $\beta = 146{,}6° + k\ .\ 180°$
   e) $\theta = 110{,}27° + k\ .\ 360°$ or
      $\theta = 249{,}73° + k\ .\ 360°$
   f) $\alpha = 210° + k\ .\ 360°$ or
      $\alpha = 330° + k\ .\ 360°$
   g) $\beta = 23{,}3° + k\ .\ 120°$
   h) $\theta = 122° + k\ .\ 180°$
   i) $\alpha = 21° + k\ .\ 180°$ or
      $\alpha = 39{,}5° + k\ .\ 90°$
   j) $\beta = 22{,}5° + k\ .\ 90°$

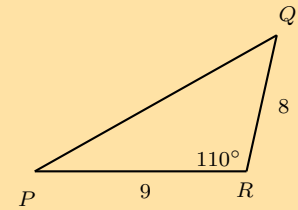
2. $\theta = 0°, 180°, 210°, 330°$ or $360°$

3.
   a) $\theta = 120° + k\ .\ 360°$ or
      $\theta = 240° + k\ .\ 360°$
   b) $\theta = 0° + k\ .\ 180°$ or
      $\theta = 146{,}3° + k\ .\ 180°$
   c) $\alpha = 36{,}9° + k\ .\ 360°$ or
      $\alpha = 143{,}1° + k\ .\ 360°$ or
      $\alpha = 216{,}9° + k\ .\ 360°$ or
      $\alpha = 323{,}1° + k\ .\ 360°$
   d) $\beta = 15° + k\ .\ 120°$ or
      $\beta = 75° + k\ .\ 120°$
   e) $\alpha = 48{,}4° + k\ .\ 180°$
   f) $\theta = 63{,}4° + k\ .\ 180°$ or
      $\theta = 116{,}6° + k\ .\ 180°$
   g) $\theta = 54{,}8° + k\ .\ 180°$ or
      $\theta = 95{,}25° + k\ .\ 180°$

4. $\beta = -70{,}5°$ or $\beta = 109{,}5°$

Exercise 6 – 10: The area rule

1. a)

Area $\triangle PQR = 17{,}5$ square units

b)

Area $\triangle PQR = 33{,}8$ square units

2. Area $\triangle XYZ = 645{,}6$ square units

3. Area $= 106{,}5$ square units

4. $\hat{C} = 72{,}2°$ or $\hat{C} = 107{,}8°$

Exercise 6 – 11: Sine rule

1. a) $\hat{P} = 92°, q = 6{,}6, p = 7{,}4$
   b) $\hat{L} = 87°, l = 1{,}3, k = 0{,}89$
   c) $\hat{B} = 76{,}8°, b = 94{,}3, c = 91{,}3$
   d) $\hat{Y} = 84°, y = 60, z = 38{,}8$

2. $\hat{B} = 32°, AB = 23, BC = 39$

3. $ST = 78{,}1$ km

4. $m = 26{,}2$

5. $BC = 3{,}2$

Exercise 6 – 12: The cosine rule

1. a) $a = 8{,}5, \hat{C} = 83{,}9°, \hat{B} = 26{,}1°$
   b) $\hat{R} = 120°, \hat{S} = 32{,}2°, \hat{T} = 27{,}8°$
   c) $\hat{M} = 27{,}7°, \hat{L} = 40{,}5°, \hat{K} = 111{,}8°$
   d) $h = 19{,}1, \hat{J} = 18{,}2°, \hat{K} = 31{,}8°$
   e) $\hat{D} = 34°, \hat{E} = 44{,}4°, \hat{F} = 101{,}6°$

2. a) $x = 4{,}4$ km
   b) $y = 63{,}5$ cm

3. a) $\hat{K} = 117{,}3°$
   b) $\hat{Q} = 78{,}5°$

Exercise 6 – 13: Area, sine and cosine rule

1. a) 7,78 km
   b) 6 km

2. $XZ = 1{,}73$ km, $XY = 0{,}87$ km

3. a) 1053 km
   b) 4,42°

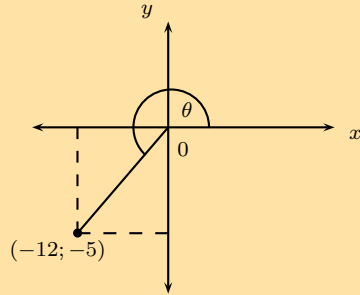
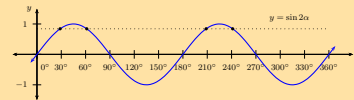
4. $DC = \frac{x \sin a \sin(b+c)}{\sin(a+c) \sin b}$

5. b) 438,5 km

6. 9,38 m^2

7. $DC = \frac{x \sin \alpha}{\sin \beta}$

Exercise 6 – 14: End of chapter exercises

1. $\sin^2 A$
2. $1\frac{1}{4}$
3. $\cos\alpha$
4. 3
7. a) -1
   b) $\theta = 135°$ or $\theta = 315°$
8. a)

   b) $-\frac{5}{13}$ and $\frac{12}{13}$
   c) $\theta = 202{,}62°$
9. a) $a = 1$ and $b = -\sqrt{3}$
   b) $-\frac{\sqrt{3}}{2}$
10. a) $x = 50{,}9°$ or $x = 309{,}1°$
    b) $x = 127{,}3°$ or $x = 307{,}3°$
    c) $x = 26{,}6°; 153{,}4° 206{,}6°$ or $333{,}4°$
11. a) $x = 55° + k\ .\ 360°$ or $x = 175° + k\ .\ 360°$
    b) $x = 180° + k\ .\ 360°$
12. a) $x = 28{,}6° + k\ .\ 180°$ or $x = 61{,}4° + k\ .\ 180°$
    b)

    c) $28{,}6°; 61{,}4°; 208{,}6°; 241{,}4°$
13. a) $A\hat{G}N = \alpha - \beta$
    b) $\hat{A} = 90° - \alpha$
    d) $H = 5$ m
14. a) $AC = 9{,}43$ m
    b) $AD = 6{,}2$ m
    c) Area $= 49{,}25$ m^2
    d) Area $= 49{,}23$ m^2

7 Measurement

Exercise 7 – 1: Area of a polygon

1. b) 240 cm
   c) $0{,}6$ m^2
   e) Wood: $233{,}2$ cm and paper: $0{,}6$ m^2
2. a) 25π units^2
   b) 20π units^2
3. a) $1{,}2$ m^2
   b) Perimeter: $414{,}8$ cm; Area $11\ 700$ cm^2
   c) 108×108cm^2

Exercise 7 – 2: Calculating surface area

1. 273 cm^2
2. Yes

Exercise 7 – 3: Calculating volume

1. a) $67{,}5$ m^2
   b) $3{,}39$ ℓ
2. b) $13{,}86$ cm
   c) $554{,}24$ m^3

Exercise 7 – 4: Finding surface area and volume

1. a) 120 cm^2
   b) 124 cm^3
   c) 40
   d) i. 120 mm
      ii. 165 mm
      iii. 589 mm

Exercise 7 – 5: The effects of k

1. a) Is halved
   b) Approx. 50 times bigger
2. a) $0{,}5W^3$
   b) $0{,}93 \times W$

Exercise 7 – 6: End of chapter exercises

2. a and d
3. a) Triangular prism
   b) Triangular pyramid
   c) Rhombic prism
4. a) i. 856 cm^2
      ii. Rectangular prism
      iii. 960 cm^3
   b) 600 cm^2
5. $\sqrt{5}x^2$
6. a) 72 000 cm^3
   b) $H = 54$ cm and $h = 60{,}2$ cm
   c) 12 732 cm^2
7. No
8. a) 10 cm × 10 cm × 10 cm
   b) 12,6 cm
9. a) Volume triples
   b) Surface area ×9
   c) Volume ×27

8 Euclidean geometry

Exercise 8 – 1: Perpendicular line from center bisects chord

1. $x = \sqrt{41}$
2. $x = \sqrt{84}$
3. $x = 10$ units
4. $TU = 2{,}66$ units
5. $x = 3{,}6$ units

Exercise 8 – 2: Angle at the centre of circle is twice angle at circumference

1. $b = 90°$
2. $c = 22{,}5°$
3. $d = 200°$
4. $e = 55°$
5. $f = 120°$

Exercise 8 – 3: Subtended angles in the same segment

1. a) $a = 21°$
   b) $c = 24°$ $d = 78°$
   c) $d = 28°$
2. a) $e = 85°$
3. $f = 35°$

Exercise 8 – 4: Cyclic quadrilaterals

1. a) $a = 93°, b = 74°$
   b) $a = 114°$
   c) $a = 29°$

Exercise 8 – 5: Tangents to a circle

1. $d = 9{,}4$ cm
2. $e = 2{,}5$ cm
3. $f = 3$ cm

Exercise 8 – 6: Tangent-chord theorem

1. a) $a = 33°, b = 33°$
   b) $c = 72°, d = 54°$
   c) $f = 38°, g = 47°$
   d) $l = 48°$
   e) $i = 40°, j = 101°, k = 40°$
   f) $m = 56°, n = 34°, o = 56°$
   g) $p = 38°, q = 52°, r = 90°$
2. $a = 26°, b = 64°, c = 128°$

Exercise 8 – 7: End of chapter exercises

1. a) $\hat{A} = x$
   b) $C\hat{O}D = 2x$
   c) $\hat{D} = 90° - x$
2. a) $\hat{D}_1 = 78°$
   b) $\hat{M}_1 = 39°$
   c) $\hat{F}_2 = 51°$
   d) $\hat{G} = 58°$
   e) $\hat{E}_1 = 32°$
3. a) $\hat{D}_2 = x$
   b) $O\hat{A}B = x$
   c) $O\hat{B}A = x$
   d) $A\hat{O}B = 180° - 2x$
   e) $\hat{C} = 90° - x$
4. a) $OM = 3$ cm
   b) $AM = 8$ cm
   c) $AB = 4\sqrt{5}$cm
5. $x = 35°$
6. a) $R\hat{Q}S, Q\hat{S}O$
   b) $P\hat{O}S = 2x$
7. a) $O\hat{D}C = 35°$
   b) $C\hat{O}D = 110°$
   c) $C\hat{B}D = 55°$
   d) $B\hat{A}D = 90°$
   e) $A\hat{D}B = 45°$
8. $x = 4y$
10. $OQ = 17$ mm
11. a) $Q\hat{R}P, Q\hat{S}P, R\hat{S}T$
    b) $S\hat{R}T = 80°, S\hat{T}R = 30°, P\hat{Q}S = 30°$
    d) $P\hat{M}Q = 110°$
13. a) $90° - \frac{x}{2}$
    b) $\frac{x}{2}$
    c) $90° - \frac{x}{2}$
14. c) $90° - 2x$
    d) $AO = 13$ cm

Exercise 9 – 1: Revision

1. R 13 630
2. R 10 246,59
3. a) R 2536
   b) R 2468,27
4. 9,38%
5. 4,56%

Exercise 9 – 2: Simple decay

1. R 112 000
2. R 941,18
3. 11,66%
4. 7 years

Exercise 9 – 3: Compound depreciation

1. R 23 766,73
2. 2229 cormorants
3. R 131 072
4. 132 221
5. 7,62 kg
6. R 85 997,13

Exercise 9 – 4: Finding i

1. 14,9%
2. 16,4%
3. 12,0%
4. 9,4%

Exercise 9 – 5: Timelines

1. R 38 588,25
2. R 35 308,00
3. R 2600
4. R 19 950,62
5. R 1 149 283,50
6. R 7359,83

Exercise 9 – 6: Nominal and effect interest rates

1. a) 12,6%
   b) 15,5%
   c) 22,1%
2. a) 16,8%; 17,7%; 17,5%
   b) 17,7%
   c) 16,8%
3. 9,1%
4. 9,4%
5. a) 9,42% is the better rate.
   b) 9,38%
   c) 9,52%

Exercise 9 – 7: End of chapter exercises

1. a) R 246 400
   b) R 265 599,87
2. R 229,92
3. a) R 8800
   b) 18,1%
4. R 238 191,17
5. a) 7,44%
   b) R 148 826,15
   c) R 135 968,69
6. a) R 8042,19
   b) 26,82%
8. a) 11,3%
   b) 11,1%
   c) 11,0%
9. R 322 580,65
10. a) 4,7%
    b) 4,8%
11. R 212 347,69

10 Probability

Exercise 10 – 1: Revision

1. $\frac{y}{r+b+y}$
2. $\frac{5}{12}$
3. $\frac{3}{8}$
4. 3
6. 5
7. a) $\frac{1}{6}$
   b) $\frac{8}{21}$
8. a) $\frac{4}{9}$
   b) $\frac{5}{9}$
   c) $\frac{1}{9}$
   d) $\frac{8}{9}$
   e) $\frac{5}{9}$
   f) $\frac{4}{9}$
9. a) $\frac{24}{116} \approx 0{,}21$
   b) $\frac{65}{116} \approx 0{,}56$
   c) 0
   d) $\frac{21}{29} \approx 0{,}72$
10. 0,28

Exercise 10 – 2: Venn diagram revision

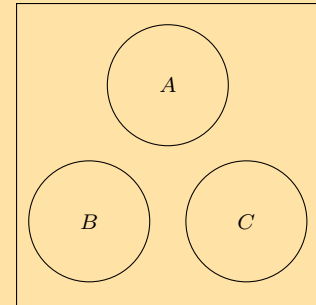
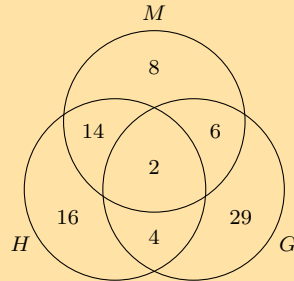
1. 0,5
2. 0,7
3. 0,18

Exercise 10 – 3: Dependent and independent events

1. a) not mutually exclusive
   b) dependent
2. a) $\frac{17}{30}$
   b) $\frac{11}{30}$
   c) Yes
   d) No
3. independent and not mutually exclusive

Exercise 10 – 4: Venn diagrams

1. a) $\frac{9}{40}$
   b) $\frac{3}{5}$
   c) $\frac{1}{10}$
   d) dependent
2. $P(Z \text{ and not } (X \text{ or } Y)) = \frac{4}{25}$
3. a)

   b) 6
   c) 29
   d) 2
4.

Exercise 10 – 5: Tree diagrams

1. $\frac{7}{36}$
2. $\frac{671}{1296}$
3. a) $\frac{1}{4}$
   b) $\frac{5}{16}$
4. a) $\frac{1}{4}$
   b) $\frac{5}{16}$

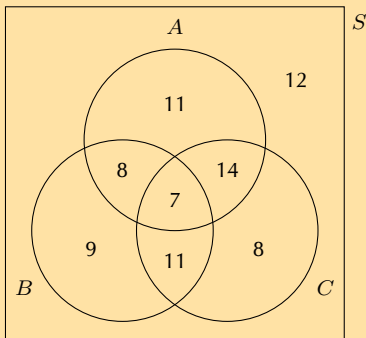
Exercise 10 – 6: Contingency tables

1. a) $\frac{5}{8}$
   b) $\frac{8}{23}$
   c) $\frac{12}{23}$
   d) dependent
2. The events are whether a bus leaves from Location A or not and whether a bus left late or not.

   The events are dependent.
3.

	A	not A	Totals
B	14	6	20
not B	21	9	30
Totals	35	15	50

Exercise 10 – 7: End of chapter exercises

1. 0,68
2. 0,24
3. a) The experiment is the outcome of selecting a particular model and colour car from the sample space of available cars.
   The sample space is {pink model A; lime-green model A; purple model A; purple model B; orange model B; multicolour model B}.
   b) 1
   c) 0
4. 0,57
5. a) 0,38
   b) 0,7
   c) 0,69
   d) 0,69
   e) 0,3
6. a) $\frac{3}{5}$
   b) $\frac{1}{2}$
   c) $\frac{1}{10}$
   d) 0
   e) $\frac{2}{5}$
   f) $\frac{9}{10}$
7. a)

   b) 12
   c) 8
   d) $\frac{17}{20}$
8. a) $h = 0{,}3$
   b) $h = 0{,}5$
9. a) 0,434
   b) 0,182
   c) 0,47
10.

$P(\text{two orange balls}) = \frac{63}{136}$

11.

	Durban	Bloemfontein	Totals
Liked living there	130	30	160
Did not like living there	140	200	340
Totals	270	230	500

The events are dependent.

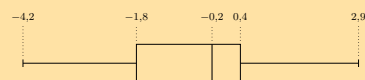
12.

	Multivitamin A	Multivitamin B	Totals
Improvement in health	400	300	700
No improvement in health	140	120	260
Totals	540	420	960

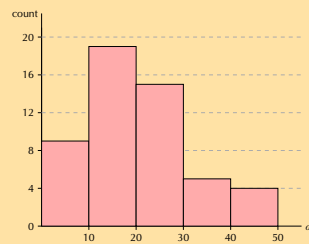
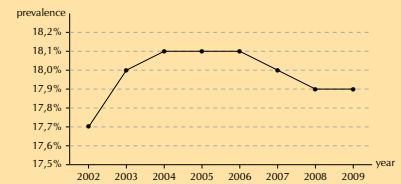
Multivitamin A is more effective.

11 Statistics

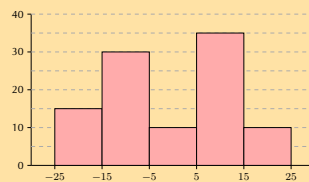
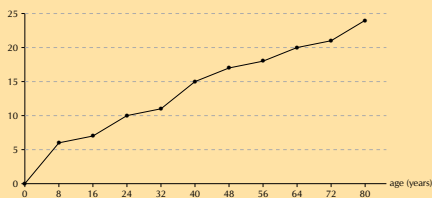
Exercise 11 – 1: Revision

1. a) mean = $-4{,}3$; first quartile = $-6{,}2$; second quartile = $-3{,}4$; third quartile = $-2{,}9$.
   b) mean = $-5{,}6$; first quartile = -60; second quartile = -6; third quartile = 65.
   c) mean = $18{,}5$; first quartile = 7; second quartile = $11{,}5$; third quartile = 33.
2. range = $9{,}6$; inter-quartile range = $2{,}51$.
3.

Exercise 11 – 2: Histograms

1. a) 170 million
   b) increasing
   c) 3 million

2.

3.

Exercise 11 – 3: Ogives

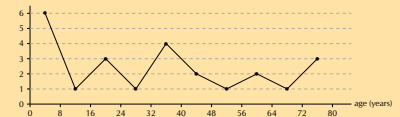
1. a) 20
   b) 15
   c) 60%

2.

3. a)

b) 11 people

c) 19 people

d) 34

e) 25,5

f)

4. e) 49,25

   f) 49,7

Exercise 11 – 4: Variance and standard deviation

1. a) Cape Town: 3,84. Durban: 3,82.
   b) Cape Town: 0,121. Durban: 0,184.
   c) Cape Town

2. Mean = 270,7. Variance = 27 435,2.

3. Mean = −1,95. Variance = 127,5.

4. a) 10,4

   b) 0,27

   c) 3

5. 13 and 20

Exercise 11 – 5: Symmetric and skewed data

1. skewed left

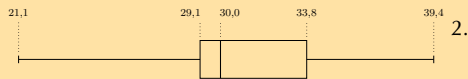
2. a) skewed right

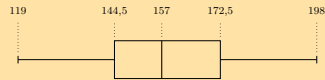
   b) skewed right

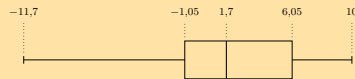
   c) skewed left

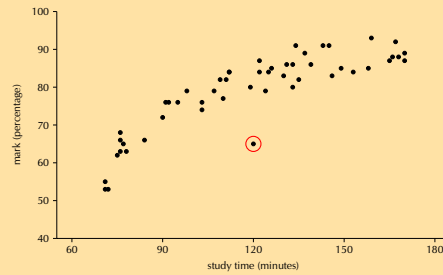
   d) symmetric

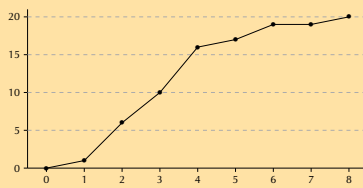
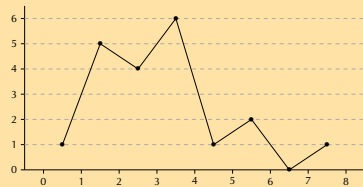
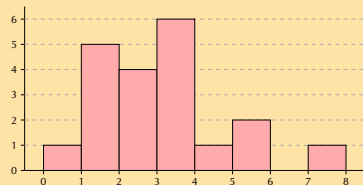
Exercise 11 – 6: Outliers

1. a)

There are two outliers on the left.

b)

There are no outliers.

c)

There is one outlier on the left.

2.

Exercise 11 – 7: End of chapter exercises

1.

2.

Skewed right

3. a) Mean $= 11\frac{3}{8}$. Standard deviation $= 6{,}69$.

b) Mean $= 5\frac{3}{8}$. Standard deviation $= 2{,}33$.

c) There are no outliers.

4. a) Mean = R 12 497,50. Standard deviation = R 1768,55.

b) 4

c) Mean = R 12 997,50. Standard deviation = R 1768,55.

d) Mean = R 13 747,25. Standard deviation = R 1945,41.

e) 4

f) 10%

5. a) 14

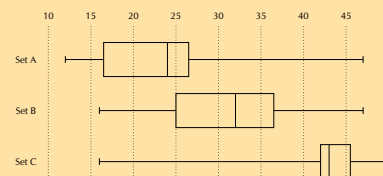
b) approximately 88 kg.

c) 75

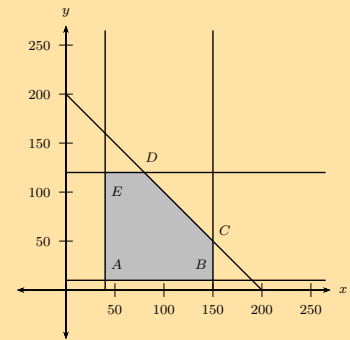
6. a) A. Mean = 23,83. Five number summary = [12 ; 16,5 ; 24 ; 26,5 ; 47].

B. Mean = 31,17. Five number summary = [16 ; 25 ; 32 ; 36,5 ; 47].

C. Mean = 41,83. Five number summary = [16 ; 42 ; 43 ; 45,5 ; 50].

b)

c) Set A: skewed left. Set B: slightly skewed left. Set C: skewed right.

Exercise 12 – 2: Optimisation

1. a) $x \leq 10$; $y \leq 10$; $x \geq 4$; $2x + 3y \leq 30$
   b) $T = 5x + 10y$
2. a) $x + y \leq 10\ 000$; $x \geq 4000$; $y \geq 2000$; $y \leq 4000$
   c) $I = 50x + 30y$
   d) R 460 000
3. a) $150x + 60y \geq 30\ 000$; $50x + 40y \geq 13\ 000$; $(10)x + (20)y \geq 5000$
   d) $E = (20\ 000)x + (10\ 000)y$
   e) 140 Super X and 150 Super Y
   f) R 4 300 000
4. a) $x \leq 300$; $y \geq 0.5x$; $x + y \leq 500$
   b) $P = (3)x + (2)y$
   c) 300 hamburgers and 200 chicken burgers
5. a) $x \leq 150; y \leq 120; x + y \leq 200; x \geq 40; y \geq 10$
   b)

   c) $P = (5)x + (10)y$
   d) 80 of card X and 120 of card Y
6. a) $4x + 3y \geq 15$; $16x + 24y \geq 72$; $x + y \leq 5$
   b) 3 packets of Vuka and 1 packets of Molo
   c) $0; 5$ or $5; 0$